A Rank-Dependent Theory for Decision under Risk and Ambiguity*

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Abstract

This paper axiomatizes, in a two-stage setup, a new theory for decision under risk and ambiguity. The axiomatized preference relation $\succeq$ on the space $\tilde{V}$ of random variables induces an ambiguity index c on the space Δ of probabilities, a probability weighting function ψ, generating the measure ν_ψ by transforming an objective probability measure, and a utility function ϕ, such that, for all $\tilde{v}, \tilde{u} \in \tilde{V}$,

$$\tilde{v} \succeq \tilde{u} \Leftrightarrow \min_{Q\in\Delta}\left\{\mathbb{E}_Q\left[\int \phi\left(\tilde{v}^{\cdot}\right)\,\mathrm{d}\nu_\psi\right] + c(Q)\right\} \geq \min_{Q\in\Delta}\left\{\mathbb{E}_Q\left[\int \phi\left(\tilde{u}^{\cdot}\right)\,\mathrm{d}\nu_\psi\right] + c(Q)\right\}.$$

Our theory extends the rank-dependent utility model of Quiggin (1982) for decision under risk to risk *and* ambiguity, reduces to the variational preferences model when ψ is the identity, and is *dual* to variational preferences when ϕ is affine in the same way as the theory of Yaari (1987) is dual to expected utility. As a special case, we obtain a preference axiomatization of a decision theory that is a rank-dependent generalization of the popular maxmin expected utility theory. We characterize ambiguity aversion in our theory.

Keywords: Risk and ambiguity; model uncertainty; robustness; dual theory; multiple priors; variational and multiplier preferences; ambiguity aversion.

JEL Classification: D81.

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1 Introduction

The distinction between risk (probabilities given) and ambiguity (probabilities unknown), after Keynes (1921) and Knight (1921), has become a central aspect in decision-making under uncertainty. Already since Ellsberg (1961) the importance of this distinction had been apparent: whereas in the classical subjective expected utility (SEU) model of Savage (1954) the distinction between risk and ambiguity was nullified through the assignment of subjective probabilities (Ramsey, 1931, de Finetti, 1931), the Ellsberg (1961) paradox showed experimentally that decisions under ambiguity could not be reconciled with any such assignment of subjective probabilities. It took, however, until the 1980s before decision models were developed that could account for ambiguity without the assignment of subjective probabilities.

Among the most popular—by now canonical—models for decision under risk and ambiguity today are maxmin expected utility (MEU, Gilboa and Schmeidler, 1989), also called multiple priors, and Choquet expected utility (CEU, Schmeidler, 1986, 1989). The former model is a decision-theoretic foundation of the classical decision rule of Wald (1950) in (robust) statistics; see also Huber (1981). Somewhat more recently, Maccheroni, Marinacci and Rustichini (2006) axiomatized the broad and appealing class of variational preferences (VP), which includes MEU and the multiplier preferences of Hansen and Sargent (2000, 2001) as special cases. Multiplier preferences have been widely used in macroeconomics, to achieve "robustness" in settings featuring model uncertainty.

In the Anscombe and Aumann (1963) setting, all the aforementioned decision models reduce to the classical Von Neumann and Morgenstern (1944) expected utility (EU) model under risk—a property that is undesirable from a descriptive perspective: it means, for example, that the Allais (1953) paradox, hence the common consequence and common ratio effects, are still present under risk; see e.g., Machina (1987).[1]

In this paper, we introduce and axiomatize, in a two-stage setup similar to the Anscombe and Aumann (1963) setting, a new theory for decision under risk and ambiguity. As we will explicate, our theory extends the rank-dependent utility (RDU) model of Quiggin (1982) for decision under risk to risk *and* ambiguity; reduces to the VP model of Maccheroni, Marinacci and Rustichini (2006) when linear in objective probabilities; and is *dual* to VP when affine in wealth in the same way as the theory of Yaari (1987)

[1]Furthermore, Machina (2009) shows that decision problems in the style of Ellsberg (1961) lead to similar paradoxes for CEU as for SEU, arising from event-separability properties that CEU retains in part from SEU.

is dual to EU for decision under risk. Thus, the theory developed in this paper may be viewed as an extension of the dual theory (DT) of Yaari (1987) and the RDU model of Quiggin (1982) for decision under risk to settings involving risk and ambiguity, just like the theory of Maccheroni, Marinacci and Rustichini (2006) is a significant extension to risk and ambiguity of the EU model for risk. As a special case, we obtain a preference axiomatization of a decision theory that is a rank-dependent generalization of the popular MEU model of Gilboa and Schmeidler (1989), and is dual to it when affine in wealth. See Table 1.[2]

Table 1: Primal, Dual and Rank-Dependent Decision Theories

	Primal	*Dual*	*Rank-Dependent*
Risk	EU (vNM, 1944)	DT (Yaari, 1987)	RDU (Quiggin, 1982)
Risk and ambiguity	MEU (GS, 1989)	***This paper***	***This paper***
Risk and ambiguity	VP (MMR, 2006)	***This paper***	***This paper***

The development of the DT of Yaari (1987) was methodologically motivated by the fact that, under EU, the decision-maker's (DM's) attitude towards wealth, as represented by the utility function, completely dictates the attitude towards risk. However, attitude towards wealth and attitude towards risk should arguably be treated separately: they are "horses of different colors" (Yaari, 1987). This is achieved within the DT and RDU models. From an empirical perspective, the DT and RDU models naturally rationalize various behavioral patterns that are inconsistent with EU. The RDU model synthesizes and encompasses both EU and DT, and serves as the main building block in prospect theory of Tversky and Kahneman (1992) discussed in much detail in Wakker (2010).[3,4]

[2]Our approach can also be used to obtain decision theories that are dual to, and a rank-dependent generalization of, CEU of Schmeidler (1989). We do not pursue this in this paper. Similar comments apply to the increasingly popular α-maxmin expected utility model and the special case of the Hurwicz expected utility model (Ghirardato, Maccheroni and Marinacci, 2004, Gul and Pesendorfer, 2015).

[3]According to Harrison and Swarthout (2016), RDU even arises as the most important non-EU model for decision under risk from a descriptive perspective.

[4]Contrary to the linearity in probabilities that occurs under EU, to which MEU and VP reduce under risk, the DT model of Yaari (1987) is affine in wealth. Yaari (1987) suggests the behavior of a profit maximizing firm as a prime example in which affineness in wealth seems particularly suitable. Other theories that stipulate affineness in wealth are provided by convex measures of risk (Föllmer and Schied, 2016, Chapter 4) encompassing many classical insurance premium principles, and by robust expectations (see e.g., Riedel, 2009, and the references therein). Despite the popularity of these theories, neither affineness in wealth nor linearity in probabilities as in EU is considered fully empirically viable for individual decision-making. Instead, we provide a general decision theory for risk and ambiguity in which preferences under risk are represented by the more general measure on the wealth-probability plane given by RDU.

Similarly, our results are both theoretically (i.e., methodologically) relevant and of potential empirical interest. At the methodological level, our theory disentangles attitude towards wealth from attitude towards risk *and* attitude towards ambiguity. We characterize (comparative) ambiguity aversion in our decision model to corroborate this separation. From an empirical perspective, an important and distinctive feature of our theory is that it accounts for ambiguity—hence is not subject to violations of subjective probabilities such as the Ellsberg paradox—and yet does not collapse to EU under risk as would be the case if the Anscombe and Aumann (1963) setting would apply—hence is not subject to the objective phenomena of the Allais paradox and related effects. We offer three additional motivations for our new decision theory in Section 5.

The numerical representation of the decision theory we axiomatize entails that the DM considers, for each random variable to be evaluated in the face of risk and ambiguity, a collection of potential probabilistic models rather than a single probabilistic model. In recent years, we have seen increasing interest in optimization, macroeconomics, finance and other fields to account for the possibility that an adopted probabilistic model is an approximation to the true probabilistic model and may be misspecified. Models that explicitly recognize potential misspecification provide a "robust" approach. Within the MEU model, the DM assigns the same plausibility to each probabilistic model in a set of probabilistic models under consideration. The multiplicity of the set of probabilistic models then reflects the degree of ambiguity. The VP model significantly generalizes the MEU model by allowing to attach a plausibility (or ambiguity) index to each probabilistic model. Such an ambiguity index also appears in the numerical representation of our decision model.

More specifically, our numerical representation U of the preference relation $\succeq$ on the space $\tilde{V}$ of random variables takes the form

$$U(\tilde{v}) = \min_{Q \in \Delta} \left\{ \mathbb{E}_Q \left[\int \phi\left(\tilde{v}^{\cdot}\right) \, \mathrm{d}\nu_\psi \right] + c(Q) \right\}, \qquad \tilde{v} \in \tilde{V}, \tag{1.1}$$

with Δ a set of probabilities on the states of the world, $c : \Delta \to [0, \infty]$ the ambiguity index, $\psi : [0,1] \to [0,1]$ a probability weighting function, ν_ψ a measure obtained by transforming an objective probability measure according to ψ, and $\phi : \mathbb{R} \to \mathbb{R}$ a utility function. Special cases of interest occur when the ambiguity index is the well-known relative entropy or Kullback-Leibler divergence (Hansen and Sargent, 2000, 2001 and Strzalecki, 2011a), or, more generally, an f-divergence measure (Csiszár, 1975, Ben-Tal, 1985, and Laeven and Stadje, 2013), or simply an indicator function that takes the value zero on a subset of Δ and ∞ otherwise. It is directly apparent from (1.1) that in the absence of uncertainty about the state of the world (i.e., in the case of risk) our decision model reduces to RDU of Quiggin (1982). The familiar utility and probability weighting functions determine the attitude towards wealth and risk. In our general model, we characterize ambiguity aversion in terms of the ambiguity index.

In essence, our axiomatization is based on a modification of two axioms stipulated by Gilboa and Schmeidler (1989) and Maccheroni, Marinacci and Rustichini (2006): the uncertainty aversion axiom and the (weak) certainty independence axiom. First, we postulate a form of ambiguity aversion (Axiom A6 below) with respect to "subjective mixtures of random variables" rather than with respect to "probabilistic mixtures of horse lotteries" as in the uncertainty aversion axiom of Gilboa and Schmeidler (1989) and Maccheroni, Marinacci and Rustichini (2006). Subjective mixtures of *horse lotteries* are due to Nakamura (1990) and Gul (1992) (see also Ghirardato and Marinacci, 2001), within a (very) different setting and for a different purpose. The subjective mixtures are employed here for *random variables*, and are extended to subjective additions of random variables.[5]

Consider two random variables with an unknown probability distribution between which the DM is indifferent. Then, our new Axiom A6 stipulates that the DM prefers a subjective mixture of the two random variables to either one in full. This constitutes a preference for *diversification*, induced by subjective mixtures of random variables with an unknown probability distribution.[6] In the primal theories of Gilboa and Schmeidler (1989) and Maccheroni, Marinacci and Rustichini (2006), uncertainty aversion instead takes the form of a preference for *randomization*. Translated to our setting of preferences over random variables, randomization stipulates that the DM prefers receiving two random variables, between which she is indifferent, with probabilities p and $1-p$, $0 < p < 1$, to obtaining one of them with certainty. Randomization as used in the primal theories provides a hedge against ambiguity by trading off ambiguity for chance; diversification as used in our theory provides a hedge against ambiguity by securing (utility units of) wealth.

Second, we replace the (weak) certainty independence axiom of Gilboa and Schmei-

[5]Subjective mixtures of horse lotteries entail that the utility profile of the subjective mixture of two horse lotteries equals the convex combination of the utility profiles of the horse lotteries themselves. Similarly, the utility profile of a subjective mixture of two random variables equals the convex combination of the utility profiles of the random variables themselves.

[6]Under affineness in wealth, this reduces to a preference for convex combinations of the two random variables.

dler (1989) and Maccheroni, Marinacci and Rustichini (2006) by a comonotonic type of independence axiom (Axiom A7 below), which pertains to subjective mixtures and subjective additions of random variables instead of probabilistic mixtures of lotteries.[7] Our approach is inspired by the "dual independence axiom" of Yaari (1987). However, in our general setting that allows for a set of probabilistic models, the implications of comonotonicity, and its interplay with ambiguity, must be reconsidered: whereas preferences over random variables may well be invariant to the (subjective) addition of comonotonic random variables when probability distributions are given (i.e., under risk) as stipulated by Yaari (1987) and Quiggin (1982), this implication may no longer be appropriate under ambiguity because such addition may impact the "level" of ambiguity (see Example 3.4 below).

Therefore, we postulate the following two versions of the dual independence axiom to extend RDU to a setting featuring risk and ambiguity: (*i*) preferences over random variables are invariant to the subjective addition of a comonotonic random variable with an objectively given probability distribution (Axiom A7); and (*ii*) preferences over random variables are invariant to subjective mixtures of the random variable and a comonotonic random variable with an objectively given probability distribution (Axiom A7^0). The former yields a decision theory that is a rank-dependent generalization of VP, the latter yields a decision theory that is a rank-dependent generalization of MEU. The mathematical details in the proofs of our characterization results are delicate.

1.1 Related Literature

In interesting and important work, Dean and Ortoleva (2017) axiomatize a decision theory that, like our theory, simultaneously allows for both violations of EU under risk à la Allais as well as violations of SEU à la Ellsberg. Their novel preference representation takes the form of a maxmin multiple priors-multiple weighting functional that distorts objective probabilities by finding the worst from a class of probability weighting functions. We briefly highlight the main differences between their work and ours. Our representation allows for a non-trivial ambiguity index, including e.g., Hansen-Sargent type robustness via Kullback-Leibler divergences, or general f-divergences. Furthermore, whereas the elements in the class of probability weighting functions in Dean and Ortoleva (2017) are all increasing and convex, our theory allows for a single, general probability

[7] As is well-known, the independence axiom and its various alternatives are key to obtaining, and empirically verifying, preference representations.

weighting function, including star-shaped or (inverse) S-shaped functions often found in experiments (see e.g., Tversky and Kahneman, 1992, Prelec, 1998 and Gonzalez and Wu, 1999) and non-decreasing functions consistent with Value-at-Risk and Expected Shortfall measures of risk (see Section 5.1). From a methodological perspective, we introduce a setting where, in the spirit of Yaari (1987), random variables, instead of sets of lotteries and acts as in Dean and Ortoleva (2017), are the basic objects. This not only fundamentally impacts the meaning of some of the axioms compared to the setting in Dean and Ortoleva (2017), but may also facilitate their interpretation. In particular, we do not stipulate preference relationships over certain *sets* of lotteries and acts but consider simple preferences over (single) random variables. Finally, our setting enables us to identify and exploit (in our technical proofs) dual relations between the primal theories given by VP and MEU and our theory (see also Table 1).[8] These dual relations constitute one of the main pillars underlying the results in this paper.

In an early version of this paper,[9] we extended Yaari's DT to risk *and* ambiguity. Then, subjective mixtures of random variables are not required; it is sufficient to consider convex combinations of the random variables themselves. The (substantially) generalized setting in the current version of the paper allows for utility functions that are non-affine in wealth, extending RDU of Quiggin (1982) to risk *and* ambiguity. In concurrent work, Wang (2022) adopts a two-stage setup that maintains the MEU (or VP) framework in the first stage and replaces EU under risk by an RDU representation in the second stage on a finite objective probability space. Whereas the numerical representation obtained in Wang (2022) is similar to that in the present version of this paper, the two approaches are (very) different, both methodologically and technically. In Wang (2022), the axioms of Gilboa and Schmeidler (1989) (or Maccheroni, Marinacci and Rustichini, 2006) over acts are maintained in the first stage. The RDU model is then induced directly in the second stage, by introducing an axiom referred to as p-trade-off consistency, which postulates that trade-offs between objective probabilities of certain ordered outcomes can be done independently of the other outcomes. Furthermore, the usual mixtures of (objective) lotteries are replaced by subjective mixtures of *probabilities*. In our approach, instead, random variables are the basic objects, leveraging duality in the sense of Yaari (1987). Our approach introduces axioms that are different, in meaning and interpretation, from those in Gilboa and Schmeidler (1989) (or Maccheroni,

[8] More specifically, our proofs employ the ('dual') space of conditional reflected quantile functions and convex duality results for niveloids.

[9] Available from `https://pure.tue.nl/ws/files/32872072/024_report.pdf`.

Marinacci and Rustichini, 2006), involving subjective mixtures of random variables and an associated diversification preference and dual independence condition, operating on a rich probability space. Thus, the two approaches rely on different behavioral axioms with their own basic objects, mixtures and spaces, and corresponding proof techniques.

1.2 Outline

This paper is organized as follows. In Section 2, we introduce our setting and notation. In Section 3, we review some preliminaries, introduce subjective mixtures of random variables and our new axioms, state our main representation results, and discuss their interpretation. Section 4 characterizes ambiguity aversion in our theory. In Section 5, we further motivate our new decision theory from three different perspectives. All proofs are relegated to the Appendix.

2 Setup and Notation

We adopt a two-stage setup as in the Anscombe and Aumann (1963) approach, with a state space that is a product space admitting a two-stage decomposition. Different from the Anscombe and Aumann (1963) model, however, we do not assume nor induce EU for risk. Our theory defines a preference relation over random variables just like Yaari (1987) for risk by which our notation is inspired. We now formalize our two-stage setup in detail.

We consider a possibly infinite set W of states of the world with σ-algebra Σ' of subsets of W that are events. We call a function $F : W \times \mathbb{R} \to [0,1]$ a conditional cumulative distribution function (CDF) if, for all $w \in W$, $F(w, \cdot)$ is a CDF and, for every $t \in \mathbb{R}$, $F(\cdot, t)$ is Σ'-measurable. Furthermore, consider, in neo-Bayesian nomenclature, an *act* or *horse (race) lottery* $f : W \to \mathcal{X}$, where $\mathcal{X}$ is an affine space of consequences. Assume in particular that $\mathcal{X}$ is given by the space of *objective lotteries*, i.e., the space of CDFs with bounded support. Then, our preference domain of random variables can be seen to correspond and be equivalent (up to a richness condition) to that of the standard Anscombe-Aumann setting as in Gilboa and Schmeidler (1989) and Maccheroni, Marinacci and Rustichini (2006), where the primitives are acts with consequences that are objective lotteries. More specifically, as we will see below, we can formally identify random variables and associated conditional CDFs with acts.

In principle, we could—as in much of the modern decision-theoretic literature—

formulate our axioms below in terms of acts. Instead, we define our axioms in terms of random variables, for clarity of exposition and ease of interpretation. The essential difference compared to Gilboa and Schmeidler (1989) and Maccheroni, Marinacci and Rustichini (2006) occurs in our Axioms A6, A7 and A8 below. Here, among other aspects, the "+" (and "$\oplus$") operation is defined *outcome-wise* instead of *probability-wise*. More specifically, in Gilboa and Schmeidler (1989) and Maccheroni, Marinacci and Rustichini (2006), the numerical representation of preferences is linear in $\mathcal{X}$ and takes the EU form under risk. Formally, in their setting, for two consequences $x_1, x_2 \in \mathcal{X}$, the mixing operation $\alpha x_1 + (1-\alpha)x_2$ represents the compound lottery that yields consequence x_1 with probability α and consequence x_2 with probability $(1-\alpha)$. By contrast, in our axioms, the "+" operation will refer to mixing the outcomes (i.e., payoffs) of the consequences x_1, x_2, meaning that, for $\alpha x_1 + (1-\alpha)x_2$, the DM receives $100 \cdot \alpha\%$ of the payoff of the consequence x_1 and $100 \cdot (1-\alpha)\%$ of the payoff of the consequence x_2. Furthermore, in our setting, mixing of payoffs will only be considered for comonotonic consequences, which move in tandem. This is most clearly expressed in terms of random variables.[10]

Consider next a non-atomic probability space (S, Σ, P), supporting a random variable U that is uniformly distributed on the unit interval under P. The product space of interest is given by $W \times S$. Assume that for every $A \in \Sigma$, the mapping $w \mapsto P[A]$ is Σ'-measurable. Let $\tilde{V}$ be the space of all bounded random variables $\tilde{v}$ defined on the space $(W \times S, \Sigma' \otimes \Sigma)$, i.e., $\tilde{v}$ is a mapping from $W \times S$ to a bounded subset of $\mathbb{R}$. Similar to Yaari (1987), realizations of the random variables in $\tilde{V}$ will be viewed as payments denominated in monetary units. For a random variable $\tilde{v} \in \tilde{V}$ and a fixed $w \in W$, the random variable $\tilde{v}^w : S \to \mathbb{R}$ is the outcome s contingent payment that the DM receives if she lives in state of the world w. This makes $\tilde{v}^w$ also interpretable as a *roulette lottery*, in neo-Bayesian nomenclature. Henceforth, $\tilde{v}^w$ and its associated roulette lottery are often identified.[11] We denote by $\tilde{V}_0$ the subspace of all random variables in $\tilde{V}$ that take only finitely many values. For fixed $w \in W$, we define the conditional CDF $F_{\tilde{v}}(w, t)$ of the Σ-measurable random variable $\tilde{v}^w$, given by $s \mapsto \tilde{v}^w(s)$, by $F_{\tilde{v}}(w, t) = P[\tilde{v}^w \leq t]$. (We sometimes omit the dependence on t, i.e., we sometimes write $F_{\tilde{v}}(w)$.) From the assumptions above, it follows that for every $t \in \mathbb{R}$, $F_{\tilde{v}}(., t)$ is Σ'-measurable.

We now identify every act $f : W \to \mathcal{X}$ with a $\Sigma' \otimes \Sigma$-measurable random variable

[10]In the setting of decision under risk, this corresponds to the dual theory of Yaari (1987).

[11]We have assumed here that in each state of the world w the possible outcomes are the same. This can simply be achieved by adding to each state of the world additional outcomes with associated probability zero.

on the product space $W \times S$, $\tilde{v} : W \times S \to \mathbb{R}$, in the following way. First, every $\tilde{v} \in \tilde{V}$ induces a conditional CDF and hence can be identified with a horse lottery by setting $f(w) := F_{\tilde{v}}(w)$. Conversely, for fixed w, every horse lottery f, given by $w \mapsto \mu^w$, for (roulette) lotteries μ^w defined on (S, Σ, P), induces a CDF $F(w, \cdot)$. Let $q(w)$ be the left-continuous inverse of $F(w)$, i.e.,

$$q(w, \lambda) = \inf\{t \in \mathbb{R} | F(w,t) \geq \lambda\}, \qquad \lambda \in (0,1).$$

Then we can define $\tilde{v}^w(s) = q(w, U(s))$ and it is easy to see that, for every $w \in W$, $\tilde{v}^w$ has the same probability distribution as $f(w)$. Hence, there is a one-to-one correspondence between equivalence classes of random variables $\tilde{v} \in \tilde{V}$ with the same conditional distributions and horse lotteries f. Thus, our preference domain of random variables on the product space $W \times S$ corresponds to the standard Anscombe-Aumann domain of acts mapping states, W, to objective lotteries of outcomes $\mathcal{X}$; we additionally require richness in the sense of a non-atomic probability space (S, Σ, P).

For some $\tilde{v} \in \tilde{V}$ (e.g., those that represent payoffs from games such as flipping coins) the DM may actually know the objective probability distribution. As in Anscombe and Aumann (1963), Gilboa and Schmeidler (1989) and Maccheroni, Marinacci and Rustichini (2006) the associated objective lotteries will play a special role for our theory. In this case, the probability distribution of $\tilde{v}$ does not depend on w, i.e., for all $w_1, w_2 \in W$, $F_{\tilde{v}}(w_1) = F_{\tilde{v}}(w_2)$. We denote the corresponding space of all the random variables in $\tilde{V}$ and $\tilde{V}_0$ that carry no ambiguity by V and V_0, respectively. For random variables $v \in V$ we usually omit the w, i.e., we just write $F_v(t)$ instead of $F_v(w,t)$. In the space V, v_n converges in distribution to v if F_{v_n} converges to F_v for all continuity points of F_v.

Furthermore, let V' be the space defined by[12]

$$V' = \left\{\tilde{v} \in \tilde{V} | \tilde{v} \text{ is independent of } s \in S, \text{ i.e., for } s_1, s_2 \in S: \ \tilde{v}^w(s_1) = \tilde{v}^w(s_2)\right\}.$$

Clearly, the space V' of all random variables in $\tilde{V}$ that carry no risk may be identified with the space of bounded measurable functions on (W, Σ'). V'_0 is defined as the corresponding subspace of bounded measurable functions that take only finitely many values.

Finally, let $\Delta(W, \Sigma')$ be the space of all finitely additive measures on (W, Σ') with mass one and denote by $\Delta_\sigma(W, \Sigma')$ the space of all probability measures on (W, Σ').

[12]For instance, for Ellsberg type urns, the space V' corresponds to payoffs depending upon outcomes from urns that contain only balls of one color, but where the specific color is unknown.

3 Representation

In this section, we provide an axiomatic foundation of a new rank-dependent theory for decision under risk and ambiguity.

3.1 Preliminaries

We define a preference relation $\succeq$ on $\tilde{V}_0$. As usual, $\succ$ stands for strict preference and $\sim$ for indifference. The preference relation $\succeq$ on $\tilde{V}_0$ induces a preference order, also denoted by $\succeq$, over random variables $s \mapsto \tilde{v}^w(s)$ through those random variables in $\tilde{V}_0$ that are associated with objective lotteries (i.e., are in V_0) by defining $\tilde{v}^w \succeq \tilde{u}^w$ if, and only if, $v \succeq u$ with $v, u \in V_0$ and $F_v(t) = F_{\tilde{v}}(w,t)$, $F_u(t) = F_{\tilde{u}}(w,t)$ for all $t \in \mathbb{R}$. (This, in turn, induces, similarly, a preference relation over monetary payments.) We assume that $\succeq$ satisfies the following properties:

Axiom *A1—Weak Order: $\succeq$ is complete and transitive. That is:*

(a) $\tilde{v} \succeq \tilde{u}$ or $\tilde{u} \succeq \tilde{v}$ for all $\tilde{v}, \tilde{u} \in \tilde{V}_0$.

(b) If $\tilde{v}, \tilde{u}, \tilde{r} \in \tilde{V}_0$, $\tilde{v} \succeq \tilde{u}$ and $\tilde{u} \succeq \tilde{r}$, then $\tilde{v} \succeq \tilde{r}$.

Whereas Yaari (1987) assumes that the preference relation is complete on the space of all random variables including those taking infinitely many values, we only assume in A1(a) that $\succeq$ is complete on the subspace $\tilde{V}_0$ of random variables that take finitely many values. We will see later that $\succeq$ and our representation results may be uniquely extended to the entire space $\tilde{V}$. For the dual theory of Yaari (1987) without the completeness axiom, see Maccheroni (2004).

Axiom *A2—Neutrality: Let $\tilde{v}$ and $\tilde{u}$ be in $\tilde{V}_0$ and have the same conditional CDFs, $F_{\tilde{v}}$ and $F_{\tilde{u}}$. Then, $\tilde{v} \sim \tilde{u}$.*

Axiom A2 states that $\succeq$ depends only on the conditional distributions. In particular, $\succeq$ induces a preference relation on the space of conditional CDFs by defining $F(\succeq)G$ if, and only if, there exist two random variables $\tilde{v}, \tilde{u} \in \tilde{V}_0$ such that $\tilde{v} \succeq \tilde{u}$ and $F_{\tilde{v}}(w) = F(w)$, $G_{\tilde{u}}(w) = G(w)$ for all $w \in W$. To simplify notation, we will henceforth use $\succeq$ both for preferences over random variables and for preferences over conditional CDFs.

Axiom *A3—Continuity: For every $v, u \in V_0$ such that $v \succ u$, and uniformly bounded sequences v_n and u_n converging in distribution to v and u, there exists an n from which* *onwards $v_n \succ u$ and $v \succ u_n$. Furthermore, for every $\tilde{v} \in \tilde{V}_0$, the sets* $\{m \in \mathbb{R} | m \succ \tilde{v}\}$ and $\{m \in \mathbb{R} | \tilde{v} \succ m\}$ *are open.*

When restricted to V_0, our continuity condition A3 is equivalent to the one employed in Yaari (1987), where it is a little stronger than that of Maccheroni, Marinacci and Rustichini (2006) or Gilboa and Schmeidler (1989).

Axiom *A4—Certainty First-Order Stochastic Dominance and Non-Degeneracy: For all $v, u \in V_0$: If $F_v(t) \leq F_u(t)$ for every $t \in \mathbb{R}$, then $v \succeq u$. Furthermore, if the inequality is sharp for some $t \in \mathbb{R}$, then $v \succ u$.*

Axiom *A5—Monotonicity: For all $\tilde{v}, \tilde{u} \in \tilde{V}_0$: If $\tilde{v}^w \succeq \tilde{u}^w$ for every $w \in W$, then $\tilde{v} \succeq \tilde{u}$.*

We postulate Axioms A1-A5 for a preference relation $\succeq$ defined on the space of finite-valued random variables $\tilde{V}_0$. However, by A2, in view of the one-to-one correspondence explicated in the previous section, it is straightforward to verify that this preference relation induces a preference relation, also denoted by $\succeq$, on the space of horse lotteries, satisfying the same axioms. Consequently, all axioms considered so far, which will be maintained in our setting, are common; see Yaari (1987), Schmeidler (1989), Gilboa and Schmeidler (1989) and Maccheroni, Marinacci and Rustichini (2006). To (strictly speaking: a subset of) the collection of axioms above, Gilboa and Schmeidler (1989) add the following two axioms (involving probabilistic mixtures):

Axiom *A6MEU—Uncertainty Aversion: If $\tilde{v}, \tilde{u} \in \tilde{V}_0$ and $\alpha \in (0,1)$, then $F_{\tilde{v}} \sim F_{\tilde{u}}$ implies $\alpha F_{\tilde{v}} + (1-\alpha) F_{\tilde{u}} \succeq F_{\tilde{v}}$.*

Axiom *A7MEU—Certainty Independence: If $\tilde{v}, \tilde{u} \in \tilde{V}_0$ and $v \in V_0$, then $F_{\tilde{v}} \succeq F_{\tilde{u}} \Leftrightarrow \alpha F_{\tilde{v}} + (1-\alpha) F_v \succeq \alpha F_{\tilde{u}} + (1-\alpha) F_v \quad$ for all $\alpha \in (0,1)$.*

With these axioms at hand, one obtains the maxmin expected utility representation, as follows:

Theorem 4.0.(i) (Gilboa and Schmeidler, 1989)
A preference relation $\succeq$ satisfies A1-A5 and A6MEU-A7MEU if, and only if, there exist an increasing and continuous[13] *function $\phi : \mathbb{R} \to \mathbb{R}$ and a non-empty, closed and convex set $C \subset \Delta(W, \Sigma')$ such that, for all $\tilde{v}, \tilde{u} \in \tilde{V}_0$,*

$$\tilde{v} \succeq \tilde{u} \Leftrightarrow \min_{Q \in C} \mathbb{E}_Q \left[\int \phi(t) F_{\tilde{v}}(., \mathrm{d}t) \right] \geq \min_{Q \in C} \mathbb{E}_Q \left[\int \phi(t) F_{\tilde{u}}(., \mathrm{d}t) \right].$$

[13] Gilboa and Schmeidler (1989) imposed slightly milder continuity and monotonicity conditions than in this paper such that in their setting ϕ was non-constant and non-decreasing.

Furthermore, $\succeq$ has a unique extension to $\tilde{V}$ satisfying the same assumptions (over $\tilde{V}$).

Somewhat more recently, Maccheroni, Marinacci and Rustichini (2006) obtained a more general representation result, which includes the maxmin expected utility representation of Gilboa and Schmeidler (1989) as a special case, but also covers the multiplier preferences employed in robust macroeconomics; see, for instance, Hansen and Sargent (2000, 2001). If, in the certainty independence axiom A7MEU, α is close to zero, then $\alpha F_{\tilde{v}} + (1-\alpha)F_v$ carries "almost no ambiguity". Hence, if a DM prefers $\tilde{v}$ to $\tilde{u}$ (as the axiom presumes), but merely because $\tilde{v}$ carries less ambiguity than $\tilde{u}$, then she may actually prefer $\alpha F_{\tilde{u}} + (1-\alpha)F_v$ to $\alpha F_{\tilde{v}} + (1-\alpha)F_v$ when α is small and ambiguity has almost ceased to be an issue. Therefore, Maccheroni, Marinacci and Rustichini (2006) suggest to replace the certainty independence axiom by the following weaker axiom:

Axiom *A7VP—Weak Certainty Independence: If $\tilde{v}, \tilde{u} \in \tilde{V}_0$, $v, u \in V_0$ and $\alpha \in (0,1)$, then $\alpha F_{\tilde{v}} + (1-\alpha)F_v \succeq \alpha F_{\tilde{u}} + (1-\alpha)F_v \Rightarrow \alpha F_{\tilde{v}} + (1-\alpha)F_u \succeq \alpha F_{\tilde{u}} + (1-\alpha)F_u$.*

Denote by $m_{\tilde{v}}$ the certainty equivalent of $\tilde{v}$, that is, $m_{\tilde{v}} \sim \tilde{v}$, $m_{\tilde{v}} \in \mathbb{R}$. Replacing A7MEU by A7VP (*ceteris paribus*) yields the following theorem:

Theorem 4.0.(ii) (Maccheroni, Marinacci and Rustichini, 2006)
A preference relation $\succeq$ satisfies A1-A5 and A6MEU-A7VP if, and only if, there exist an increasing and continuous[14] function $\phi : \mathbb{R} \to \mathbb{R}$ and a grounded,[15] convex and lower-semicontinuous function $c : \Delta(W, \Sigma') \to [0, \infty]$ such that, for all $\tilde{v}, \tilde{u} \in \tilde{V}_0$,

$$\tilde{v} \succeq \tilde{u} \;\Leftrightarrow\; \min_{Q \in \Delta(W,\Sigma')} \left\{ \mathbb{E}_Q \left[\int \phi(t) F_{\tilde{v}}(., \mathrm{d}t) \right] + c(Q) \right\} \geq \min_{Q \in \Delta(W,\Sigma')} \left\{ \mathbb{E}_Q \left[\int \phi(t) F_{\tilde{u}}(., \mathrm{d}t) \right] + c(Q) \right\}.$$

Furthermore, for each ϕ there exists a unique minimal c_0 given by

$$c_0(Q) = \sup_{v' \in V_0'} \left\{ m_{\phi(v')} - \mathbb{E}_Q[\phi(v')] \right\}.$$

3.2 Subjective Mixtures of Random Variables

In this subsection, we introduce and analyze *subjective mixtures of random variables*. Let us, following the insightful Ghirardato, Maccheroni, Marinacci and Siniscalchi (2003),

[14]Maccheroni, Marinacci and Rustichini (2006), just like Gilboa and Schmeidler (1989), imposed slightly weaker continuity and monotonicity conditions than in the present paper, such that ϕ was non-constant and non-decreasing.

[15]We say that c is *grounded* if $\min_{Q \in \Delta(W,\Sigma')} c(Q) = 0$.

first recall preference averages and subjective mixtures for the class of biseparable preferences, which was introduced and axiomatized by Ghirardato and Marinacci (2001).

For $A \in \Sigma$, define $t\,A\,x := t\,I_A(s) + x\,I_{A^c}(s)$. Here, $t, x \in \mathbb{R}$, I_A is the indicator of event A and A^c is the complement of A. Let U be a standard uniform random variable on the probability space associated with the second stage and define the set $E_y := \{y < U < 1-y\}$, $y \in [0, 1/2]$. For $t > x$, $v_y := t\,E_y\,x = t\,I_{E_y} + x\,I_{E_y^c}$ converges to t as y tends to 0 and to x as y tends to $1/2$. By continuity as assumed in A3,[16] there must then be a $y^* \in (0, 1/2)$ such that $t \succ t\,E_{y^*}\,x \succ x$. Any event E satisfying this preference ordering for some $t > x$ will in the sequel be called an *essential event*.

We call our binary relation $\succeq$ on V_0 a *biseparable preference* if it has a nontrivial representation $\mathcal{U} : V_0 \to \mathbb{R}$ for which:

(i) there exists $\rho : [0, 1] \to [0, 1]$ with ρ not identical zero or one on $(0, 1)$ such that, for all $t \succeq x$ and all $A \in \Sigma$,

$$\mathcal{U}(t\,A\,x) = \phi(t)\rho(P(A)) + \phi(x)\,(1 - \rho(P(A)))\,,$$

where $\phi(t) \equiv \mathcal{U}(t)$ for all $t \in \mathbb{R}$ is increasing;

(ii) $\mathcal{U}(\mathbb{R})$ is convex.

Averages of, say, t and x with respect to a symmetric function $h : \mathbb{R}^2 \to \mathbb{R}$ can often be defined as the element z such that $h(t, x) = h(z, z)$, with the arithmetic and geometric average corresponding to $h(t, x) = t + x$ and $h(t, x) = t \cdot x$. Now given $t, x \in \mathbb{R}$, if $t \succeq x$ we say that $y \in \mathbb{R}$ is a *preference average* of t and x (given an essential event E) if $t \succeq y \succeq x$ and

$$t\,E\,x \sim m_{tEt}\,E\,m_{xEx} \sim m_{tEy}\,E\,m_{yEx}.$$

Next, fix some event $B \in \Sigma$, and then construct *state by state* a random variable in V_0 such that every state s yields the certainty equivalent of the bet $v(s)\,B\,u(s)$ for $v, u \in V_0$. That is, the *state-wise B-mixture* of v and u is the random variable $\mathrm{sub}(vBu) \in V_0$ formally defined as follows: For all $s \in S$,

$$\mathrm{sub}(vBu)(s) \equiv m_{(v(s)Bu(s))}.$$

[16]Let u be the midpoint between t and x. Define $y^* := \inf\{y | v_y \succ u\}$. The set over which the infimum is taken is not empty because, by A4, $y = 0$ is in this set. As by A3 the set $M = \{v | u \succ v\}$ is open, $v_{y^*} = \lim_n v_{y^*+1/n} \in M^c$ cannot be in M, and thus $v_{y^*} \succeq u$. Suppose that $v_{y^*} \succ u$. Then, by the definition of y^*, we would have that $v_{y^*} \succ u \succeq v_{y^*-1/n}$ and by passing to the limit using A3 arrive at a contradiction. Hence, $v_{y^*} \sim u$.

Provided that v dominates u state-wise, the constructed random variable yields state-wise indifference to bets on the event B; the random variable thus constructed can via CDFs be identified with a subjective mixture of lotteries.

Subsequently, we say that two random variables $\tilde{v}, \tilde{u} \in \tilde{V}$ are *comonotonic* if, for every $w \in W$ and every $s, s' \in S$,

$$(\tilde{v}^w(s') - \tilde{v}^w(s))(\tilde{u}^w(s') - \tilde{u}^w(s)) \geq 0.$$

Comonotonic random variables do not provide hedging potential because their realizations move in tandem without generating offsetting possibilities (Schmeidler, 1986, 1989, Yaari, 1987).

The next axiom is a weak version of the independence axiom of EU for binary comonotonic random variables, discussed in Ghirardato and Marinacci (2001) (for horse lotteries). Extending Chew and Karni (1994), it weakens in particular Schmeidler's comonotonic independence axiom, and can be traced back to Gul (1992) and Nakamura (1990).

AXIOM *A8—Binary Comonotonic Independence:*[17] *For every essential $A \in \Sigma$, every $B \in \Sigma$, and for all $v, u, r \in V_0$ and $x, x', x'', y, y', y'' \in \mathbb{R}$ such that $v = x\,A\,y$, $u = x'\,A\,y'$ and $r = x''\,A\,y''$, if v, u, r are pairwise comonotonic, and $x, x' \geq x''$ and $y, y' \geq y''$ (or $x'' \geq x, x'$ and $y'' \geq y, y'$), then*

$$v \succeq u \Longrightarrow \mathrm{sub}(vBr) \succeq \mathrm{sub}(uBr).$$

The following proposition is suitably adapted from Ghirardato and Marinacci (2001), Theorem 11, and Ghirardato, Maccheroni, Marinacci and Siniscalchi (2003), Proposition 1:

Proposition 3.1 *Suppose that A1-A4 and A8 hold. Then $\succeq$ is a biseparable preference on V_0, where ρ is unique and ϕ is continuous and unique up to a positive affine transformations. Furthermore, for $t, x \in \mathbb{R}$ and each essential event $E \in \Sigma$, $y \in \mathbb{R}$ is a preference average of t and x given E if and only if*

$$\phi(y) = \frac{1}{2}\phi(t) + \frac{1}{2}\phi(x).$$

Hence, preference averages of t and x given E exist for every essential event $E \in \Sigma$, and

[17] In Ghirardato and Marinacci (2001), this axiom is referred to as "Binary Comonotonic Act Independence". As in our setting it is imposed on random variables (rather than acts, i.e., horse lotteries), we omit the term "act".

they are independent of the choice of E and of the normalization of $\phi(\cdot)$.

If $z \in \mathbb{R}$ is a preference average of t and x, then, by the proposition above, we may define y as a preference average of t and z satisfying $\phi(y) = (3/4)\,\phi(t) + (1/4)\,\phi(x)$. Using standard preference continuity arguments, we may also define $\alpha : (1-\alpha)$-utility mixtures, $\alpha \in [0,1]$. *Subjective mixtures of random variables* on the space $\tilde{V}_0$ can then be defined point-wise: Given $\tilde{v}, \tilde{u}$ and $\alpha \in [0,1]$, the *subjective mixture* $\alpha\tilde{v} \oplus (1-\alpha)\tilde{u}$ is the random variable $\tilde{r}$ defined by

$$\tilde{r}(w) := \alpha\tilde{v}(w) \oplus\, (1-\alpha)\tilde{u}(w), \tag{3.1}$$

for any $w \in W$, where $\tilde{r}$ satisfies

$$\phi(\tilde{r}(w)) = \alpha\phi(\tilde{v}(w)) + (1-\alpha)\phi(\tilde{u}(w)). \tag{3.2}$$

Next, let us define *subjective additions* of random variables. We first state the following definition.

Definition 3.2 *We call z a preference doubling of x (given an essential event E) and write $z = 2 \otimes x$ if, in case $x \geq 0$, we have $z \succeq x$ and $z\,E\,0 \sim m_{zEx}\,E\,m_{xE0}$; or, in case $x < 0$, we have $x \succeq z$ and $0\,E\,z \sim m_{0Ex}\,E\,m_{xEz}$.*

Proposition 3.3 *$z = 2 \otimes x$ if, and only if,*

$$\frac{1}{2}\phi(z) + \frac{1}{2}\phi(0) = \phi(x). \tag{3.3}$$

In particular, z is invariant under positive affine transformations of the function ϕ, and for all functions ϕ with $\phi(0) = 0$, we have $\phi(z) = 2\phi(x)$.

Finally, we define point-wise

$$\tilde{v}(w) \oplus \tilde{u}(w) := 2 \otimes (\frac{1}{2}\tilde{v}(w) \oplus \frac{1}{2}\tilde{u}(w)), \tag{3.4}$$

and note that, for any ϕ from Proposition 3.1 with $\phi(0) = 0$,

$$\phi(\tilde{v}(w) \oplus \tilde{u}(w)) = \phi(\tilde{v}(w)) + \phi(\tilde{u}(w)). \tag{3.5}$$

3.3 New Axioms

We replace the uncertainty aversion axiom of Gilboa and Schmeidler (1989) and Maccheroni, Marinacci and Rustichini (2006) (Axiom A6MEU) by the following assumption:

Axiom *A6—Dual Ambiguity Aversion: If $v', u' \in V_0'$ and $\alpha \in (0,1)$, then $v' \sim u'$ implies $\alpha v' \oplus (1-\alpha)u' \succeq v'$.*

Observe that, different from Axiom A6MEU (as well as Axioms A7MEU and A7VP) that applies to mixtures of conditional CDFs, Axiom A6 considers subjective mixtures of random variables, introduced in Section 3.2. By using "$\oplus$" rather than "+", the utility profiles rather than the random variables themselves are combined; the DM takes subjective mixtures of random variables that carry no risk (are in V_0') and that she is indifferent to. A dual ambiguity averse DM, then, prefers the "diversified", convex combination of the utility profiles of the two random variables ($\alpha v' \oplus (1-\alpha)u'$) to the original non-diversified random variable (v' or u'). Note that, as a special case when $\phi = \text{id}$, "$\oplus$" corresponds to "+". Hence, one can anticipate that the preference for diversification induced by A6 stems from attitude towards ambiguity rather than from attitude towards wealth. In fact, consistent with experiments, our theory does not assume a specific attitude towards wealth such as a globally concave ϕ.

Before stating our new Axiom A7, we briefly discuss "dual independence", which inspired it. Yaari (1987), in the context of decision under risk, asserts that a preference of v to u induces a preference of $\alpha v + (1-\alpha)r$ to $\alpha u + (1-\alpha)r$, $\alpha \in (0,1)$, in case v, r and u, r are *pairwise comonotonic* (pc). (This assertion also implies (*ceteris paribus*) a preference of $v+r$ to $u+r$.) In particular, Yaari (1987) replaces the independence axiom of EU by the following assumption (involving outcome mixtures of random variables), restricted to decision under risk:

Axiom *A7D—Dual Independence: Let $v, u, r \in V_0$ and assume that v, r and u, r are pc. Then, for every $\alpha \in (0,1)$, $v \succeq u \Rightarrow \alpha v + (1-\alpha)r \succeq \alpha u + (1-\alpha)r$.*

To extend this axiom to subjective mixtures of random variables, "+" in A7D would have to be replaced by "$\oplus$".

Against this background, first consider $\tilde{v}, \tilde{u}, \tilde{r} \in \tilde{V}_0$ and suppose that the DM prefers $\tilde{v}$ to $\tilde{u}$. Is it natural to require that the DM then also prefers $\tilde{v} \oplus \tilde{r}$ to $\tilde{u} \oplus \tilde{r}$, or $\alpha\tilde{v} \oplus (1-\alpha)\tilde{r}$ to $\alpha\tilde{u} \oplus (1-\alpha)\tilde{r}$ with $\alpha \in (0,1)$, in general (without comonotonicity imposed)? If $\tilde{u}$ and $\tilde{r}$ are not comonotonic, then the DM may try to employ $\tilde{r}$ to hedge against adverse realizations of $\tilde{u}$. As a result, $\tilde{u} \oplus \tilde{r}$ can conceivably be "better diversified" than $\tilde{v} \oplus \tilde{r}$

(depending on the joint stochastic nature of $\tilde{u}, \tilde{r}$ on the one hand and that of $\tilde{v}, \tilde{r}$ on the other), and the DM may instead prefer $\tilde{u} \oplus \tilde{r}$ to $\tilde{v} \oplus \tilde{r}$.

Hence, consider $\tilde{v}, \tilde{u}, \tilde{r} \in \tilde{V}_0$, suppose that $\tilde{v} \succeq \tilde{u}$, as before, and suppose furthermore that $\tilde{v}, \tilde{r}$ and $\tilde{u}, \tilde{r}$ are pc. Should a DM then also prefer $\tilde{v} \oplus \tilde{r}$ to $\tilde{u} \oplus \tilde{r}$? (Note that this is not implied by Axiom A7D, which requires the random variables to live in the space V_0, and besides considers "+" instead of "$\oplus$".) Even though (subjectively) adding $\tilde{r}$ does, in view of the pc assumption, not induce any discriminatory hedging potential, it may still impact the ambiguity "level", in a discriminatory manner, leading to a preference reversal.

Consider the following example, where for ease of exposition, we assume an affine function ϕ so that "$\oplus$" and "+" agree:

Example 3.4 Consider two urns, A and B, and 50 balls, 25 of which are red and 25 of which are black. Every urn contains 25 balls. The exact number of balls per color in each urn is unknown. Furthermore, consider two urns, C and D, and 50 balls, 30 of which are red and 20 of which are black. As for A and B, every urn contains 25 balls, but the exact number of balls per color in each urn is unknown.

Denote by p_i the (unknown) probability of drawing a red ball from urn $i, i \in \{A, B, C\}$. Draw a random number, say U, from the set $\{1, \ldots, 25\}$ with each number having the same likelihood. Consider:

(i) the random variable $\tilde{v}$ that represents a payoff of \$100 if U is smaller than or equal to the number of red balls in urn C, and 0 otherwise;

(ii) the random variable $\tilde{u}$ that represents a payoff of \$100 if U is smaller than or equal to the number of red balls in urn A, and 0 otherwise;

(iii) the random variable $\tilde{r}$ that represents a payoff of \$100 if U is smaller than or equal to the number of red balls in urn B, and 0 otherwise.

Note that $\tilde{v}, \tilde{r}$ and $\tilde{u}, \tilde{r}$ are pc. Typically, $\tilde{v} \succeq \tilde{u}$ because $30 > 25$. At the same time, the DM may prefer $\tilde{u} + \tilde{r}$ to $\tilde{v} + \tilde{r}$, because the former is, loosely speaking, less ambiguous than the latter. More specifically, the unknown probability of drawing red from A is connected (complementary) to the unknown probability of drawing red from B: with certainty, $p_A + p_B = 1$. By contrast, the probability of drawing red from B (or A) is not connected to the probability of drawing red from C. Mathematically, $\tilde{u} + \tilde{r}$ yields at least \$100 with probability $\max\{p_A, 1 - p_A\} \geq 50\%$, and it yields exactly \$200 with probability $\min\{p_A, 1-p_A\} = 1-\max\{p_A, 1-p_A\}$. On the other hand, $\tilde{v}+\tilde{r}$ has potential realizations \$0, \$100, and \$200 with unknown probabilities, where no non-trivial upper or lower bounds can be given. $\triangledown$

We will assert that, if $\tilde{v}, \tilde{u}, \tilde{r} \in \tilde{V}_0$, $\tilde{v} \succeq \tilde{u}$, and $\tilde{v}, \tilde{r}$ and $\tilde{u}, \tilde{r}$ are pc, then the implication $\tilde{v} \oplus \tilde{r} \succeq \tilde{u} \oplus \tilde{r}$ only holds if $\tilde{r}$ carries no ambiguity (i.e., is in V_0), hence cannot impact the ambiguity level, in a discriminatory manner. This motivates to replace the weak certainty independence axiom by the following assumption:

Axiom *A7—Weak Certainty Dual Independence: Let $\tilde{v}, \tilde{u} \in \tilde{V}_0$ and $r \in V_0$. Suppose that $\tilde{v}, r$ and $\tilde{u}, r$ are pc. Then, $\tilde{v} \succeq \tilde{u} \Rightarrow \tilde{v} \oplus r \succeq \tilde{u} \oplus r$.*

3.4 Main Results

The following axiomatic characterization generalizes the rank-dependent utility model to the setting of risk *and* ambiguity. The result may also be viewed as a generalization of the variational preferences model; see Table 1. As there are no concavity restrictions on the utility function, and no convexity restrictions on the probability weighting function, the result encompasses (inverse) S-shaped utility and probability weighting functions used in prospect theory and supported by empirical evidence.

Let $\psi : [0,1] \to [0,1]$ be a non-decreasing and continuous function satisfying $\psi(0) = 0$ and $\psi(1) = 1$. We refer to ψ as a probability weighting or distortion function. For $v \in V$, we define the measure ν_ψ through

$$\int v \,\mathrm{d}\nu_\psi := \int_{-\infty}^{0} (\psi(1 - F_v(t)) - 1)\,\mathrm{d}t + \int_{0}^{\infty} \psi(1 - F_v(t))\,\mathrm{d}t.$$

One readily verifies that, for $a > 0$ and $b \in \mathbb{R}$, $\int (av + b)\,\mathrm{d}\nu_\psi = a \int v \,\mathrm{d}\nu_\psi + b$. We now state our main result, which provides a representation theorem characterizing a preference relation satisfying Axioms A1-A8:

Theorem 3.5 (α) *A preference relation $\succeq$ satisfies A1-A8 if, and only if, there exist a non-constant, non-decreasing and continuous function $\psi : [0,1] \to [0,1]$ with $\psi(0) = 0$ and $\psi(1) = 1$, an increasing and continuous function $\phi : \mathbb{R} \to \mathbb{R}$ and a grounded, convex and lower-semicontinuous function $c : \Delta(W, \Sigma') \to [0,\infty]$ such* *that, for all* $\tilde{v}, \tilde{u} \in \tilde{V}_0$,

$$\tilde{v} \succeq \tilde{u} \Leftrightarrow \min_{Q \in \Delta(W,\Sigma')} \left\{ \mathbb{E}_Q \left[\int \phi(\tilde{v}^{\cdot}) \, \mathrm{d}\nu_\psi \right] + c(Q) \right\} \\ \geq \min_{Q \in \Delta(W,\Sigma')} \left\{ \mathbb{E}_Q \left[\int \phi(\tilde{u}^{\cdot}) \, \mathrm{d}\nu_\psi \right] + c(Q) \right\}. \tag{3.6}$$

Furthermore, for each ϕ *there exists a unique minimal* $c_{\min}$ *satisfying* (3.6) *given by*

$$c_{\min}(Q) = \sup_{v' \in V_0'} \left\{ m_{\phi(v')} - \mathbb{E}_Q[\phi(v')] \right\}.$$

(β) *There exists a unique extension of* $\succeq$ *to* $\tilde{V}$ *satisfying A1-A8 on* $\tilde{V}$ *and* (3.6).

(Here, $\tilde{v}^{\cdot}$ denotes the random variable given by $s \mapsto \tilde{v}^{\cdot}(s)$.[18])

A natural question is whether it is possible to restrict the class of *penalty* functions on $\Delta(W, \Sigma')$ to more specific ones, such as a penalty function that only takes the values zero or infinity, and what this would entail behaviorally. To this end, we replace (*ceteris paribus*) Axiom A7 by the following stronger (i.e., more restrictive) assumption:

Axiom *A7*^{0}*—Certainty Dual Independence: Let* $\tilde{v}, \tilde{u} \in \tilde{V}_0$ *and* $r \in V_0$*. Suppose that* $\tilde{v}, r$ *and* $\tilde{u}, r$ *are pc. Then,* $\tilde{v} \succeq \tilde{u} \;\Leftrightarrow\; \alpha\tilde{v} \oplus (1-\alpha)r \succeq \alpha\tilde{u} \oplus (1-\alpha)r$ *for all* $\alpha \in (0, 1)$.

If Axioms A1-A6, A7^0 and A8 hold, then we obtain the rank-dependent generalization of the popular Gilboa and Schmeidler (1989) maxmin expected utility representation:

Theorem 3.6 *(a) A preference relation* $\succeq$ *satisfies A1-A6, A7*^{0} *and A8 if, and only if, there exist a non-constant, non-decreasing and continuous function* $\psi : [0, 1] \to [0, 1]$ *with* $\psi(0) = 0$ *and* $\psi(1) = 1$*, an increasing and continuous function* $\phi : \mathbb{R} \to \mathbb{R}$ *and a non-empty, closed and convex set* $\mathcal{Q} \subset \Delta(W, \Sigma')$ *such that, for all* $\tilde{v}, \tilde{u} \in \tilde{V}_0$,

$$\tilde{v} \succeq \tilde{u} \Leftrightarrow \min_{Q \in \mathcal{Q}} \mathbb{E}_Q \left[\int \phi(\tilde{v}^{\cdot}) \, \mathrm{d}\nu_\psi \right] \geq \min_{Q \in \mathcal{Q}} \mathbb{E}_Q \left[\int \phi(\tilde{u}^{\cdot}) \, \mathrm{d}\nu_\psi \right]. \tag{3.7}$$

Furthermore, there exists a unique extension of $\succeq$ *to* $\tilde{V}$ *satisfying A1-A6, A7*^{0} *and A8 on* $\tilde{V}$ *and* (3.7).

[18] In fully explicit form, the numerical representation in (3.6) may thus be expressed as

$$\min_{Q \in \Delta(W,\Sigma')} \left\{ \int_W \int_S \phi(\tilde{v}^w(s)) \, \nu_\psi(\mathrm{d}s) \, Q(\mathrm{d}w) + c(Q) \right\}.$$

(b) If moreover the numerical representation in (3.7) is continuous from below, then $\mathcal{Q} \subset \Delta_\sigma(W, \Sigma')$, i.e., the minimum may be taken over a convex set of probability measures.

3.5 Interpretation

Define U as the numerical representation in (3.6), i.e.,

$$U(\tilde{v}) = \min_{Q \in \Delta(W,\Sigma')} \left\{ \mathbb{E}_Q \left[\int \phi(\tilde{v}^{\boldsymbol{\cdot}}) \, \mathrm{d}\nu_\psi \right] + c(Q) \right\}.$$

The numerical representation may be given the following interpretation. The function c is non-negative and grounded, i.e., for every $Q \in \Delta(W, \Sigma')$, $c(Q) \geq 0$, and there exists at least one measure $P' \in \Delta(W, \Sigma')$ such that $c(P') = 0$. This measure P' may be considered to be the DM's candidate model (i.e., "approximation"), selected from the set of all measures on (W, Σ'). If the DM believes that P' is a good (reliable) candidate model, then she can simply take the (P'-)expectation over all evaluations of an objective lottery, i.e., calculate the rank-dependent $\mathbb{E}'[\int \phi(\tilde{v}^{\boldsymbol{\cdot}}) \, \mathrm{d}\nu_\psi]$, which would correspond to $c(Q) = \infty$ if $Q \neq P'$, $Q \in \Delta(W, \Sigma')$, in (3.6). In many situations, however, the DM may not fully trust his candidate model P' and takes other measures on (W, Σ') into account.

One way to proceed would be to assume a worst case approach and consider the representation $\min_{Q \in \Delta(W,\Sigma')} \mathbb{E}_Q[\int \phi(\tilde{v}^{\boldsymbol{\cdot}}) \, \mathrm{d}\nu_\psi] = \min_w \int \phi(\tilde{v}^w) \, \mathrm{d}\nu_\psi$, which corresponds to $c(Q) = 0$ for all $Q \in \Delta(W, \Sigma')$, as in Theorem 3.6. In this case, the DM would consider all measures on (W, Σ') equally plausible. Alternatively, the DM may consider his candidate model P' to be more plausible than other measures, but still wants to take other measures into account (non-trivially). In this case, she would take the minimum over all measures in $\Delta(W, \Sigma')$, and "penalize" every measure Q not equal to P' by a penalty $c(Q)$. This penalty depends on the degree of plausibility that the DM associates to the measure Q. The function c is therefore often referred to as an ambiguity index. Such procedures that explicitly account for the fact that the measure P' is only an approximation and may deviate from the true measure are often referred to as robust approaches. They are robust against "malevolent nature".

In statistics, engineering and optimal control, risk management, and robust macroeconomics, the plausibility of the measure Q is often expressed by the relative entropy of Q with respect to the approximation P'; see Csiszár (1975), Ben-Tal (1985), Hansen and Sargent (2000, 2001), Maccheroni, Marinacci and Rustichini (2006), Strzalecki (2011a) and Laeven and Stadje (2013, 2014). In our setting, this would lead to $c(Q) = \theta R(Q|P')$ with $R(Q|P') = \mathbb{E}_Q\left[\log\left(\frac{\mathrm{d}Q}{\mathrm{d}P'}\right)\right]$ and θ a non-negative constant. The relative entropy, also referred to as Kullback-Leibler divergence, measures the deviation of Q from P' and is zero if and only if $Q \equiv P'$. Thus, measures that are close to P' are penalized weakly, while measures that deviate strongly from P' are penalized strongly. Specifically, for $v' \in V'_0$,

$$\min_{Q\in\Delta(W,\Sigma')} \{\mathbb{E}_Q[v'] + \theta R(Q|P')\} = -\theta \log \{\mathbb{E}'[\exp(-v'/\theta)]\}.$$

In general, for $\tilde{v} \in \tilde{V}_0$,

$$\min_{Q\in\Delta(W,\Sigma')} \left\{\mathbb{E}_Q\left[\int \phi(\tilde{v}^{\cdot})\,\mathrm{d}\nu_\psi\right] + c(Q)\right\} = -\theta \log\left(\mathbb{E}'\left[\exp\left(\int -\phi(\tilde{v}^{\cdot})\,\mathrm{d}\nu_\psi/\theta\right)\right]\right).$$

Other ways of penalizing "deviating beliefs" include $c(Q) = \theta G(Q|P')$ with $G(Q|P')$ the relative Gini index given by $\mathbb{E}\left[\left(\frac{\mathrm{d}Q}{\mathrm{d}P'} - 1\right)^2\right]$. Again, $G(Q|P') = 0$ if and only if $Q = P'$; $G(Q|P')$ measures how much the ratio of Q and P' deviates from one; see e.g., Maccheroni, Marinacci, Rustichini and Taboga (2004). Maccheroni, Marinacci and Rustichini (2006) also propose to weight every state of the world w by a weighting function $h : W \to \mathbb{R}_+$ satisfying $\int_W h(w)P'(\mathrm{d}w) = 1$. The penalty functions are then given by $c(Q) = \int_W h(w) \log\left(\frac{\mathrm{d}Q}{\mathrm{d}P'}(w)\right) Q(\mathrm{d}w)$ and $c(Q) = \int_W h(w)\left(\frac{\mathrm{d}Q}{\mathrm{d}P'}(w) - 1\right)^2 Q(\mathrm{d}w)$.[19]

4 Probabilistic Sophistication and Ambiguity Aversion

4.1 Probabilistic Sophistication

We refer to P' as a *reference measure* on (W, Σ') if the DM is indifferent between random variables that have the same probability distribution under P'. We say that a DM who adopts a reference measure on (W, Σ') is *probabilistically sophisticated*; see Machina and Schmeidler (1992) and Epstein (1999). Our axioms do not (necessarily) imply the existence of a reference measure on (W, Σ'). But in case there is a reference measure P' on (W, Σ'), we define, for a given $v' \in V'_0$, $F'_{v'}$ by $F'_{v'}(t) = P'[v' \leq t]$. With slight abuse of notation we say that $v' \succeq_2 u'$ if, for every $t \in \mathbb{R}$, $\int_{-\infty}^t F'_{v'}(\tau)\,\mathrm{d}\tau \leq \int_{-\infty}^t F'_{u'}(\tau)\,\mathrm{d}\tau$. We call $\succeq_2$ *second order stochastic dominance* (SSD) on V' with respect to P'; see

[19] Note that these two penalty functions are not *probabilistically sophisticated* on (W, Σ') unless $h \equiv 1$; see also Section 4.

Rothschild and Stiglitz (1970). More generally, we say that $v' \succeq_{\phi,2} u'$ if, for every $t \in \mathbb{R}$, $\int_{-\infty}^{t} F'_{\phi(v')}(\tau)\,\mathrm{d}\tau \leq \int_{-\infty}^{t} F'_{\phi(u')}(\tau)\,\mathrm{d}\tau$ and call $\succeq_{\phi,2}$ *ϕ-second order stochastic dominance* (ϕ-SSD) on V' with respect to P'. The following proposition shows that the availability of a non-atomic reference measure P' is equivalent to requiring that the DM respects ϕ-SSD on V'_0:[20,21]

Proposition 4.1 *Suppose that a preference relation $\succeq$ satisfies A1-A8. Then the following statements are equivalent:*

(a) there exists a non-atomic reference measure, say P', on (W, Σ').

(b) $\succeq$ respects ϕ-SSD on V'_0 with respect to a non-atomic P'.

Furthermore, if ϕ is concave, $\succeq$ admitting a (non-atomic) reference measure P' is equivalent to $\succeq$ respecting SSD on V'_0.

4.2 Ambiguity Aversion

Subsequently, we say that $\succeq$ is *more ambiguity averse* than $\succeq^*$ if, for all $\tilde{v} \in \tilde{V}_0$ and $v \in V_0$,

$$\tilde{v} \succeq v \Rightarrow \tilde{v} \succeq^* v.$$

Similar definitions of *comparative ambiguity aversion* can be found, for instance, in Epstein (1999), Ghirardato and Marinacci (2002) and Maccheroni, Marinacci and Rustichini (2006); see also the early Yaari (1969), Schmeidler (1989) and Gilboa and Schmeidler (1989).[22] Our notion of more ambiguity averse agrees with the comparative ambiguity aversion concept in Ghirardato and Marinacci (2002) and Maccheroni, Marinacci and Rustichini (2006).

The following result characterizes comparative ambiguity aversion in the setting of our main representation result:

[20] Recall that P' is non-atomic if the probability space (W, Σ') is rich enough to support a uniformly distributed random variable.

[21] The preference relation $\succeq$ respects ϕ-SSD on V'_0 if, for all $v', u' \in V'_0$ with $v' \succeq_{\phi,2} u'$, $v' \succeq u'$.

[22] The difference between definitions of uncertainty aversion consists primarily in the "factorization" of ambiguity attitude and risk attitude. Schmeidler (1989) and Gilboa and Schmeidler (1989) adopt the Anscombe-Aumann framework with objective unambiguous lotteries. Epstein (1999), by contrast, instead of adopting a two-stage setup and assuming that there exists a space of objective lotteries, models ambiguity by assuming that there exists a set of events $\mathcal{A}$ that *every DM* considers to be unambiguous. Then he defines comparative ambiguity aversion through the random variables that are measurable with respect to $\mathcal{A}$. The model-free factorization approach of Ghirardato and Marinacci (2002) in principle encompasses both approaches to modeling ambiguity.

Proposition 4.2 *Consider two preference relations, $\succeq$ and $\succeq^*$, induced by assuming Axioms A1-A8. Then, $\succeq$ is more ambiguity averse than $\succeq^*$ if, and only if, $\succeq^*$ and $\succeq$ may be identified with (ϕ^*, ψ^*, c^*) and (ϕ, ψ, c) such that $\phi^* = \phi$, $\psi^* = \psi$ and $c^* \geq c$.*

We note that in the absence of a probability weighting function, Proposition 4.2 holds similarly in the primal framework of Maccheroni, Marinacci and Rustichini (2006).

In Epstein (1999), Ghirardato and Marinacci (2002) and Maccheroni, Marinacci and Rustichini (2006) a DM is considered to be *ambiguity averse* if and only if she is more ambiguity averse than an *ambiguity neutral* DM. While in Ghirardato and Marinacci (2002) and Maccheroni, Marinacci and Rustichini (2006) ambiguity neutrality is equivalent to having SEU preferences, Epstein (1999) identifies ambiguity neutrality with probabilistic sophistication. Ghirardato and Marinacci (2002), however, argue that in full generality (unless the probability space is rich enough) probabilistically sophisticated behavior may still include behavior that can be considered to be ambiguity averse.[23] Consequently, in our setting, instead of identifying ambiguity neutrality ($\succeq^{\text{AN}}$) with probabilistic sophistication, it seems more natural to define $\succeq^{\text{AN}}$ via a numerical representation that induces computing a plain expectation on the space W with respect to some measure P'. In other words, we consider a DM to be *ambiguity neutral* if there exist a measure P', a utility function ϕ and a probability weighting function ψ such that, for all $\tilde{v}, \tilde{u} \in \tilde{V}_0$,

$$\tilde{v} \succeq \tilde{u} \Leftrightarrow \mathbb{E}_{P'}\left[\int \phi(\tilde{v}^{\cdot})\, \mathrm{d}\nu_{\psi}\right] \geq \mathbb{E}_{P'}\left[\int \phi(\tilde{u}^{\cdot})\, \mathrm{d}\nu_{\psi}\right].$$

Next, we say that a DM with a preference relation $\succeq$ is *ambiguity averse* if there exists an ambiguity neutral DM with a preference relation $\succeq^{\text{AN}}$ such that $\succeq$ is more ambiguity averse than $\succeq^{\text{AN}}$.

Proposition 4.3 *If $\succeq$ satisfies A1-A8, then $\succeq$ is ambiguity averse.*

[23]For instance, if W has only finitely many elements, identifying ambiguity neutrality with probabilistic sophistication would imply that a DM with a numerical representation of the form $U(\tilde{v}) = \min_{Q\in\Delta(W,\Sigma')}\{\mathbb{E}_Q[\int \phi(\tilde{v})\, \mathrm{d}\nu_\psi]\} = \inf_w \int \phi(\tilde{v}^w)\, \mathrm{d}\nu_\psi$ would be ambiguity neutral, at least, if P' does not exclude any $w \in W$. (That is, $P'[w] > 0$ for all $w \in W$.) This seems counterintuitive in our setting, since the "worst ambiguity case" possible is assumed. A worst case DM is also probabilistically sophisticated if W is a subset of $\mathbb{R}^d$ and $P' \sim Leb$. $W \subset \mathbb{R}^d$ is typically satisfied in a Bayesian framework. Strzalecki (2011b), however, proves that in the specific framework of Maccheroni, Marinacci and Rustichini (2006), ambiguity neutrality in the sense of Epstein (1999), with non-trivial no-ambiguity sets, implies that the DM has preferences given by SEU. Marinacci (2002) had proven the same result under MEU.

5 Motivation and Applications

Besides the Allais and Ellsberg paradoxes, which can jointly be rationalized by our theory, and the disentanglement of attitudes towards ambiguity, risk, and wealth that our theory permits, we offer three additional motivations for our rank-dependent theory for decision under risk and ambiguity.

5.1 Robust Risk Management

Our theory provides a decision-theoretic foundation for robust tail risk measures, unifying tail risk measures as described e.g., in Föllmer and Schied (2016), Section 4.6, and models with robustness of e.g., Hansen and Sargent (1995, 2001, 2007) type or general f-divergences. Indeed, our theory encompasses tail risk measures when probabilities are given, but also accounts for ambiguity.

Tail risk measures are extensively used by financial institutions and regulators to control and manage risks and to determine adequate capital reserves. The industry-standard tail risk measures are Value-at-Risk (VaR_λ) and Expected Shortfall (ES_λ). Value-at-Risk is the amount of capital needed to guarantee that with a certain probability no shortfall will be suffered over a pre-specified time horizon.[24] Expected Shortfall measures the expected loss beyond the Value-at-Risk. More formally, for a random variable v with a given probabilistic model P, $\mathrm{VaR}_\lambda(v) := \inf\{t \in \mathbb{R} | P[-v \leq t] \geq 1-\lambda\}$, $\lambda \in (0,1)$, and $\mathrm{ES}_\lambda(v) := \frac{1}{\lambda}\int_0^\lambda \mathrm{VaR}_\gamma(v)\,\mathrm{d}\gamma$, $\lambda \in (0,1]$.

As is well-known (see, e.g., Föllmer and Schied, 2016, Section 4.6), the numerical representation of Yaari's DT can be expressed as

$$U(v) = \int_{-\infty}^{0} \left(\psi(P[v > t]) - 1\right)\,\mathrm{d}t + \int_0^\infty \psi(P[v > t])\,\mathrm{d}t = \int_0^1 \mathrm{VaR}_\gamma(v)\,\mathrm{d}\psi(1-\gamma), \quad (5.1)$$

provided either VaR_λ or ψ is continuous. Thus, DT corresponds to "weighted VaR" risk measures, where quantiles are weighted according to the probability weighting function ψ. Expected Shortfall corresponds to (minus) weighted VaR with probability weighting function given by $\psi(\gamma) = \frac{1}{\lambda}\max(\gamma - (1-\lambda), 0)$.[25] Clearly, weighted VaR risk measures are measures of "tail risk".

[24] That is, it is the quantile function of the respective probability distribution; see, e.g., Duffie and Pan (1997) and Jorion (1997) for a detailed discussion.

[25] Other examples of weighted VaR risk measures include the inter-quartile range, the absolute deviation, Gini-related risk measures, Range Value-at-Risk, and dual-power risk measures; see Denneberg (1994), Föllmer and Schied (2016), Eeckhoudt and Laeven (2022) and the references therein.

Weighted VaR and related risk measures have been widely considered in economics, finance, insurance, operations research and statistics.[26] The primal theories for decision under risk and ambiguity of Table 1, however, do not provide a foundation for the use of weighted VaR risk measures under a known probabilistic model: a DM who complies with one of the primal theories adopts the EU model under risk—evaluating the expected utility of the risk or computing the corresponding certainty equivalent or the indifference price. None of these formally correspond to using a weighted VaR risk measure.

In recent years—in particular after the failure of risk management systems during the global financial crisis—, it has been argued that by assuming a known probabilistic model, risk measures such as weighted VaR fail to take into account that the adopted probabilistic model may be misspecified: incorrect and only an approximation. Consequently, there have been calls to seek for risk measures that account for ambiguity and lead to more robust risk management, by ensuring that policy performs well under a wide range of potential probabilistic models; see, e.g., Hansen (2014). Such robust risk measures arise in much generality in our representation (1.1), taking ϕ to be affine.

For example, robust VaR risk measures with robustness of max-min type, Hansen-Sargent type (i.e., with Kullback-Leibler divergence/relative entropy) or with general f-divergences occur as special cases of our numerical representation, as follows:

$$U(\tilde{v}) = \min_{Q \in \Delta} \left\{ \mathbb{E}_Q \left[\mathrm{VaR}_\lambda \left(\tilde{v}^{\cdot} \right) \right] + c(Q) \right\}.$$

Risk measures in this spirit have recently been used e.g., by Ghaoui, Oks and Oustry (2003), Embrechts, Puccetti, and Rüschendorf (2013), Li (2018), Cai, Li and Mao (2023) and Pesenti, Wang and Wang (2022). We provide their decision-theoretic foundation. Indeed, contrary to the primal theories, our new theory for decision under risk and ambiguity provides a decision-theoretic, axiomatic foundation for such an approach: it unifies weighted VaR risk measures and the robust approach that explicitly takes ambiguity into account.

5.2 Portfolio Choice with Mean Risk Models

An important, related motivation for our theory is that it provides a decision-theoretic foundation for the use of *mean risk* performance criteria in optimal portfolio choice. In general, mean risk models are not compatible with the primal theories of Table 1, but

[26] See, e.g., Ruszczyński and Vanderbei (2003), Cherny (2006), Cherny and Madan (2009), Basak and Shapiro (2001), Kou, Peng and Heyde (2013), Eeckhoudt, Laeven and Schlesinger (2020).

have seen much interest in theory and applications.

More specifically, letting $\tilde{v}$ denote the uncertain return of a portfolio at time T where $T > 0$ is a fixed time horizon, Ruszczyńsky and Vanderbei (2003), Wozabal (2014), He, Jin and Zhou (2015) and Cai, Li and Mao (2023), among many others, consider performance criteria of the form

$$\mathbb{E}_P[\tilde{v}] - \rho(\tilde{v}) = \mathbb{E}_P[\tilde{v}] - \min_{Q\in\Delta}\left\{\mathbb{E}_Q\left[\int \tilde{v}^{\cdot}\, \mathrm{d}\nu_\psi\right] + c(Q)\right\}, \tag{5.2}$$

where $\phi = \mathrm{id}$ and, in the case of mean-(robust) Expected Shortfall, $\psi(\gamma) = \frac{1}{\lambda}\max(\gamma-(1-\lambda), 0)$. In particular, $\rho(\tilde{v})$ is a coherent risk measure in Ruszczyńsky and Vanderbei (2003) and a robustified convex risk measure in Wozabal (2014), both with special emphasis given to (robustified) Expected Shortfall, while it is a weighted VaR risk measure in He, Jin and Zhou (2015) and a robustified weighted VaR risk measure in Cai, Li and Mao (2023). These approaches, and, importantly, the diverse optimal investment behavior induced by them, are all compatible with our theory. That is, a DM maximizing (5.2) can be identified with a DM choosing her optimal portfolio under preferences given by (1.1). Thus, again, our representation provides the decision-theoretic foundation.

Weighted VaR risk measures, with convex ψ, are special cases of coherent (and convex) risk measures, which have attracted considerable attention in the literature.[27] Via their dual representation, coherent risk measures can be given the interpretation of accounting for ambiguity. More specifically, a coherent risk measure can be identified with a DM who computes the expectation of a random variable with respect to a probabilistic model, but is uncertain about the probabilistic model, and therefore considers a robust, worst-case expectation over a collection of probabilistic models. That is, the DM is *risk* neutral but averse to *ambiguity*. For law-invariant coherent risk measures, such as Expected Shortfall, this interpretation, while formally correct, may however be unnatural: when a DM calculates Expected Shortfall, the probability distribution of the random variable is often assumed to be given, hence the DM faces decision under risk rather than under ambiguity, and displays risk aversion. The different possible interpretations of coherent risk measures arise from the fact that coherent risk measures operate in a one-stage setting, where attitudes towards risk are not distinguished from attitudes to-

[27]See e.g., Artzner et al. (1999), Carr, Geman and Madan (2001), Föllmer and Schied (2002), Frittelli and Rosazza Gianin (2002), Dana (2005), Ruszczyński and Shapiro (2006), Lesnevski, Nelson and Staum (2007), Laeven and Stadje (2013, 2014), Föllmer and Schied (2016) and the references therein. Specifically, modulo a change of sign, weighted VaR risk measures with convex ψ are coherent risk measures that are law-invariant and additive for comonotonic random variables; see Kusuoka (2001).

wards ambiguity. Our decision theory encompasses coherent and convex risk measures as special cases while maintaining a clear separation between attitude towards risk and attitude towards ambiguity.

5.3 Preference for Randomization vs. Preference for Diversification

An important aspect of our theory is that one of its key axioms entails a preference for *diversification*, while the corresponding axiom in the primal theories is based on *randomization*. In particular, our Axiom A6 stipulates that if a DM is indifferent between two portfolios A and B with unknown probability distributions, then holding a subjective mixture of $\alpha\%$ of A and $(100-\alpha)\%$ of B is preferred to holding only A or only B. The primal theories, instead, assert that the DM prefers a probabilistic mixture where she receives portfolio A with $\alpha\%$ probability and portfolio B with $(100-\alpha)\%$ probability to having for sure only A or only B.

A preference for diversification is a trait of behavior that is widespread in financial, insurance, and other real-life situations, often summarized as "Don't put all your eggs in one basket". It can be traced back to Bernoulli (1738), is a main principle in economics, operations research, and statistics, and is at the basis of classical optimal investment and asset pricing theories such as Mean-Variance optimization and the Capital Asset Pricing Model; see, e.g., Berger and Eeckhoudt (2021) for a detailed discussion. These authors also show that the value of diversification may reduce under ambiguity aversion. That is, existing definitions of ambiguity aversion, and the primal theories of Table 1, do not necessarily induce a preference for diversification. Instead, our Axiom A6 directly implies a diversification preference.

I Appendix: Proofs

Proof of Proposition 3.3. Assume without loss of generality that $x \geq 0$. Then,

$$\begin{aligned}
\mathcal{U}\left(m_{zEx}\, E\, m_{xE0}\right) &= \phi(m_{zEx})\rho(P(E)) + \phi(m_{xE0})\left(1-\rho(P(E))\right) \\
&= \phi(z)(\rho(P(E)))^2 + 2\phi(x)\left(1-\rho(P(E))\right)\rho(P(E)) \\
&\quad + \phi(0)\left(1-\rho(P(E))\right)^2 \\
&= \phi(z)\rho(P(E)) + \phi(0)\left(1-\rho(P(E))\right),
\end{aligned}$$

where the last equality holds if, and only if, (3.3) holds. $\square$

Proof of Theorem 3.5. For showing "⇒", we let the function ϕ be defined through Proposition 3.1, whereas for showing "⇐", the function ϕ is given by the thesis of Theorem 3.5, in particular, by representation (3.6). By adding or subtracting a constant, we may in both cases assume that $\phi(0) = 0$. By re-scaling ϕ and c if necessary, we may additionally assume that $\phi(1) = 1$. Denote by ϕ^{-1} the (increasing and continuous) inverse of the (increasing and continuous) function ϕ. Since ϕ is increasing and continuous, the image of ϕ, henceforth denoted by $\mathrm{Im}(\phi) \subset \mathbb{R}$, is an interval. Let

$$\tilde{V}_0^{\phi} := \{\tilde{v} \in \tilde{V}_0 | \tilde{v} \text{ only takes values in } \mathrm{Im}(\phi)\},$$

and define V_0^{ϕ} and $V_0^{\prime,\phi}$ similarly. Clearly, all these spaces are convex as $\mathrm{Im}(\phi)$ is a convex set.

Next, for $\tilde{v}, \tilde{u} \in \tilde{V}_0^{\phi}$, we define:[28]

$$\tilde{v} \succeq^* \tilde{u} \text{ if, and only if, } \phi^{-1}(\tilde{v}) \succeq \phi^{-1}(\tilde{u}). \tag{I.1}$$

We state the following lemma:

Lemma I.1 *$\succeq^*$ satisfying A1-A7 with "$\oplus$" (in A6-A7) replaced by "$+$" and all axioms restricted to random variables taking values in Im(ϕ) (i.e., with $\tilde{V}_0, V_0, V_0', m \in \mathbb{R}$ replaced by $\tilde{V}_0^{\phi}, V_0^{\phi}, V_0^{\prime,\phi}, m \in Im(\phi)$) is equivalent to $\succeq$ satisfying A1-A7.*

Proof. For A1-A5, the equivalence is straightforward to see, noting for A3 that by continuity of ϕ and ϕ^{-1}, $\phi(A)$ and $\phi^{-1}(B)$ are open sets if $A \subset \mathbb{R}$ and $B \subset \mathrm{Im}(\phi)$ are open. Let us show the equivalence for A6 and A7.
A6: $\succeq$ "⇒" $\succeq^*$. Assume that A6 holds for $\succeq$ and let us show that A6 also holds for $\succeq^*$. By definition of $\succeq^*$,

$$v' \sim^* u' \text{ if, and only if, } \phi^{-1}(v') \sim \phi^{-1}(u').$$

From A6 for $\succeq$, we then have that $\tilde{r} := \alpha\phi^{-1}(v') \oplus (1-\alpha)\phi^{-1}(u') \succeq \phi^{-1}(v')$. Hence, by definition (3.1)–(3.2),

$$\alpha v' + (1-\alpha)u' = \phi(\tilde{r}) \succeq^* \phi(\phi^{-1}(v')) = v'.$$

[28]The auxiliary notation $\succeq^*$ is not to be confused with the preference relation of the less ambiguity averse DM in Section 4.2.

A7: $\succeq$ "$\Rightarrow$" $\succeq^*$. Recall (I.1). Now $r \in V_0^\phi$ being comonotonic to $\tilde{v}$ and $\tilde{u}$ is equivalent to $\phi^{-1}(r)$ being comonotonic to $\phi^{-1}(\tilde{v})$ and $\phi^{-1}(\tilde{u})$. Hence, (I.1) and A7 entail that

$$\tilde{r} := \phi^{-1}(\tilde{v}) \oplus \phi^{-1}(r) \succeq \phi^{-1}(\tilde{u}) \oplus \phi^{-1}(r) =: \tilde{\tilde{r}}.$$

Thus, by definition (3.4)–(3.5),

$$\tilde{v} + r = \phi(\tilde{r}) \succeq^* \phi(\tilde{\tilde{r}}) = \tilde{u} + r.$$

A6: $\succeq$ "$\Leftarrow$" $\succeq^*$. From $v' \sim u'$ it follows that $\phi(v') \sim^* \phi(u')$ and therefore $\alpha\phi(v') + (1 - \alpha)\phi(u') \succeq^* \phi(v')$. Thus,

$$\alpha v' \oplus (1-\alpha)u' = \phi^{-1}(\alpha\phi(v') + (1-\alpha)\phi(u')) \succeq \phi^{-1}(\phi(v')) = v'.$$

A7: $\succeq$ "$\Leftarrow$" $\succeq^*$. Clearly, $\phi(r)$ is comonotonic to $\phi(\tilde{v})$ and $\phi(\tilde{u})$. It follows by A7 for $\succeq^*$ that $\phi(\tilde{v}) + \phi(r) \succeq^* \phi(\tilde{u}) + \phi(r)$. Then we can conclude that

$$\tilde{v} \oplus r = \phi^{-1}(\phi(\tilde{v}) + \phi(r)) \succeq \phi^{-1}(\phi(\tilde{u}) + \phi(r)) = \tilde{u} \oplus r.$$

$\square$

Upon exploiting Lemma I.1 jointly with Proposition 3.1, we can prove Theorem 3.5 by establishing the following result:

Theorem I.2 *(α) A preference relation $\succeq^*$ satisfies A1-A7 with "$\oplus$" (in A6-A7) replaced by "+" and for random variables taking values in Im(ϕ) (i.e., with $\tilde{V}_0, V_0, V_0', m \in \mathbb{R}$ replaced by $\tilde{V}_0^\phi, V_0^\phi, V_0'^{,\phi}, m \in$ Im(ϕ)) if, and only if, there exist a non-constant, non-decreasing and continuous function $\psi : [0,1] \to [0,1]$ with $\psi(0) = 0$ and $\psi(1) = 1$ and a grounded, convex and lower-semicontinuous function $c : \Delta(W, \Sigma') \to [0, \infty]$ such that, for all $\tilde{v}, \tilde{u} \in \tilde{V}_0^\phi$,*

$$\tilde{v} \succeq^* \tilde{u} \Leftrightarrow \min_{Q\in\Delta(W,\Sigma')} \left\{ \mathbb{E}_Q\left[\int \tilde{v}^{\cdot}\, \mathrm{d}\nu_\psi \right] + c(Q) \right\} \geq \min_{Q\in\Delta(W,\Sigma')} \left\{ \mathbb{E}_Q\left[\int \tilde{u}^{\cdot}\, \mathrm{d}\nu_\psi \right] + c(Q) \right\}. \tag{I.2}$$

Furthermore, there exists a unique minimal $c_{\min}$ satisfying (I.2) given by

$$c_{\min}(Q) = \sup_{v'\in V_0'^{,\phi}} \{ m_{v'} - \mathbb{E}_Q[v'] \},$$

where $m_{v'}$ is the certainty equivalent of v' under $\succeq^*$.

(*β*) *There exists a unique extension of $\succeq^*$ to $\tilde{V}^\phi$ satisfying A1-A7 on $\tilde{V}^\phi$ and* (I.2), *with "$\oplus$" in A6-A7 replaced by "$+$".*

Proof of Theorem I.2. Using for A7 that the integral w.r.t. $\mathrm{d}\nu_\psi$ in (I.2) is additive for comonotonic random variables, the only property that is not straightforward to verify in the "if" part of Theorem I.2(α) is the continuity property (Axiom A3). Let U be the numerical representation in (I.2). This implies that, for all $v \in V_0^\phi$,

$$U(v) = \int_{-\infty}^{0} (\psi(1 - F_v(t)) - 1)\,\mathrm{d}t + \int_0^\infty \psi(1 - F_v(t))\,\mathrm{d}t.$$

The first part of A3 would follow if we could show that U is continuous with respect to weak convergence of uniformly bounded sequences. Thus, suppose that v_n is a uniformly bounded sequence in V_0^ϕ, and $v_n \to v$, in distribution. Then, by definition, F_{v_n} converges to F_v at all continuity points of F_v. Because F_v and the F_{v_n}'s are non-decreasing functions, they are continuous, Lebesgue almost everywhere. But this implies that F_{v_n} converges to F_v, Lebesgue almost everywhere. Furthermore, because v_n is uniformly bounded by a constant, say M, $F_{v_n}(t) \in \{0, 1\}$ for $t \notin [-M, M]$. In view of the point-wise convergence of F_{v_n} to F_v, Lebesgue almost everywhere, this implies that $F_v(t) \in \{0, 1\}$ for $t \notin [-M, M]$, as well. Finally, because ψ is a continuous function, it is bounded on $[0, 1]$. Hence,

$$\begin{aligned}
\lim_n U(v_n) &= \lim_n \int_{-\infty}^{0} (\psi(1 - F_{v_n}(t)) - 1)\,\mathrm{d}t + \int_0^\infty \psi(1 - F_{v_n}(t))\,\mathrm{d}t \\
&= \lim_n \int_{-M}^{0} (\psi(1 - F_{v_n}(t)) - 1)\,\mathrm{d}t + \int_0^M \psi(1 - F_{v_n}(t))\,\mathrm{d}t \\
&= \int_{-M}^{0} (\psi(1 - F_v(t)) - 1)\,\mathrm{d}t + \int_0^M \psi(1 - F_v(t))\,\mathrm{d}t = U(v),
\end{aligned}$$

as desired. Proving the second part of Axiom A3 is straightforward and is therefore omitted, as is the verification of Axioms A1-A2 and A4-A7.

The proof of the "only if" part of Theorem I.2(α) consists of the following four steps:

1. We first show that $\succeq^*$ has a numerical representation U on $\tilde{V}_0^\phi$ satisfying certain properties.

2. Next, we prove that, for all $v \in V_0^\phi$, $U(v) = \int v\,\mathrm{d}\nu_\psi$.

3. Then, we show that, for all $v' \in V_0'^{,\phi}$,

$$U(v') = \min_{Q \in \Delta(W,\Sigma')} \{\mathbb{E}_Q[v'] + c(Q)\}. \tag{I.3}$$

4. Finally, we derive from Steps 2 and 3 that (I.2) holds on $\tilde{V}_0^\phi$.

Before proceeding to Step 1, we state the following preliminary lemmata, assuming Axioms A1-A7 hold:

Lemma I.3 *Let $\tilde{v}, \tilde{u}, \tilde{v}+m, \tilde{u}+m \in \tilde{V}_0^\phi$ and $m \in \mathbb{R}$. If $\tilde{v} \succ^* \tilde{u}$ and $\tilde{v}, \tilde{u}$ are pc, then $\tilde{v}+m \succ^* \tilde{u}+m$.*

Proof. By A7, $\tilde{v}+m \succeq^* \tilde{u}+m$. Suppose that $\tilde{v}+m \sim^* \tilde{u}+m$ would hold. Then, again by A7, $\tilde{v} = \tilde{v}+m-m \sim^* \tilde{u}+m-m = \tilde{u}$, which is a contradiction. □

Lemma I.4 *For every $\tilde{v} \in \tilde{V}_0^\phi$ there exists a certainty equivalent $m^*_{\tilde{v}} \in Im(\phi)$ such that $\tilde{v} \sim^* m^*_{\tilde{v}}$.*

Proof. Suppose, by contradiction, that the lemma does not hold. Then, the two sets $\{m \in \mathrm{Im}(\phi)\,|m \succ^* \tilde{v}\}$ and $\{m \in \mathrm{Im}(\phi)\,|\tilde{v} \succ^* m\}$ are disjoint, open (by A3) and their union is equal to $\mathrm{Im}(\phi)$. Furthermore, the two sets are non-empty since, for example, $\max \tilde{v} \in \mathrm{Im}(\phi)$ and $\mathrm{Im}(\phi)$ is an open set (as the image of an open set of a strictly increasing function), so that for an $\epsilon > 0$ small enough $\max \tilde{v} + \epsilon \in \mathrm{Im}(\phi)$. Hence, by A4,

$$m := \max \tilde{v} + \epsilon \succ^* \max \tilde{v} \succeq^* \tilde{v},$$

so that $\{m \in \mathrm{Im}(\phi)\,|m \succ^* \tilde{v}\}$ is indeed non-empty. Similarly, the set $\{m \in \mathrm{Im}(\phi)\,|\tilde{v} \succ^* m\}$ can be seen to be non-empty. Because the union of two non-empty disjoint open sets cannot be equal to an open interval (namely $\mathrm{Im}(\phi)$), this leads to a contradiction. Thus, there exists $m^*_{\tilde{v}} \in \mathbb{R}$ such that $\tilde{v} \sim^* m^*_{\tilde{v}}$. Since $\min \tilde{v} \leq \tilde{v} \leq \max \tilde{v}$ and $\min \tilde{v}, \max \tilde{v} \in \mathrm{Im}(\phi)$ (which is an interval), it follows that $m^*_{\tilde{v}} \in \mathrm{Im}(\phi)$. □

Step 1:

We prove first that $\succeq^*$ has a numerical representation $U : \tilde{V}_0^\phi \to \mathbb{R}$, i.e., for all $\tilde{v}, \tilde{u} \in \tilde{V}_0^\phi$,

$$\tilde{v} \succeq^* \tilde{u} \Leftrightarrow U(\tilde{v}) \geq U(\tilde{u}).$$

We will further show that U satisfies the following properties:

(i) *Conditional Law Invariance:* $U(\tilde{v})$ depends only on $F_{\tilde{v}}$.

(ii) *Continuity:* Suppose that v_n is a uniformly bounded sequence in V_0^ϕ converging in distribution to $v \in V_0^\phi$, then $\lim_n U(v_n) = U(v)$.

(iii) *Certainty First-Order Stochastic Dominance:* For all $v, u \in V_0^\phi$: If $F_v(t) \leq F_u(t)$ for every $t \in \mathbb{R}$, then $U(v) \geq U(u)$.

(iv) *Monotonicity:* For all $\tilde{v}, \tilde{u} \in \tilde{V}_0^\phi$: If $U(\tilde{u}^w) \leq U(\tilde{v}^w)$ for every $w \in W$, then $U(\tilde{u}) \leq U(\tilde{v})$.

(v) *Certainty Comonotonic Additivity:* Let $\tilde{v} \in \tilde{V}_0^\phi$ and $r \in V_0^\phi$. Suppose that $\tilde{v}, r$ are pc and $\tilde{v} + r \in \tilde{V}_0^\phi$. Then $U(\tilde{v} + r) = U(\tilde{v}) + U(r)$.

(vi) *Certainty Positive Homogeneity:* For all $v \in V_0^\phi$ and $\lambda \geq 0$, $U(\lambda v) = \lambda U(v)$ provided that $\lambda v \in V_0^\phi$.

(vii) *Translation Invariance:* For all $\tilde{v} \in \tilde{V}_0^\phi$ and $m \in \mathbb{R}$ such that $\tilde{v} + m \in \tilde{V}_0^\phi$, $U(\tilde{v} + m) = U(\tilde{v}) + m$.

(viii) *Ambiguity Concavity*: If $v', u' \in V_0^{\prime,\phi}$ and $\alpha \in (0,1)$, then $U(\alpha v' + (1-\alpha)u') \geq \alpha U(v') + (1-\alpha)U(u')$.

Assume Axioms A1-A7 hold. For $\tilde{v} \in \tilde{V}_0^\phi$, set $U(\tilde{v}) = m^*_{\tilde{v}}$. By Lemma I.4, U is well-defined and, by A4, $m^*_{\tilde{v}}$ is unique. Note that with this definition, for all $m \in \text{Im}(\phi)$, $U(m) = m$. Furthermore, it follows from the strict monotonicity (A4) that $U(\tilde{v}) > U(\tilde{u})$ if and only if $\tilde{v} \succ^* \tilde{u}$. Thus, U is a numerical representation of $\succeq^*$.

Next, let us show that U satisfies properties (i)-(viii). Properties (i)-(iv) with all random variables involved only taking values in $\text{Im}(\phi)$ follow directly from Axioms A1-A5 and the fact that U is a numerical representation of $\succeq^*$.

To see property (vii), note that Axiom A7 implies that if $\tilde{v}, \tilde{u}, \tilde{v} + m, \tilde{u} + m \in \tilde{V}_0$, we have

$$\tilde{v} \sim^* \tilde{u} \Leftrightarrow \tilde{v} + m \sim^* \tilde{u} + m. \tag{I.4}$$

We claim that this implies that U is translation invariant. This can be seen as follows. If $\tilde{v} \sim^* m^*_{\tilde{v}}$, then, for all $m \in \mathbb{R}$ such that $\tilde{v} + m \in \tilde{V}_0^\phi$,

$$\min \tilde{v} + m \leq m^*_{\tilde{v}} + m \leq \max \tilde{v} + m.$$

Since $\tilde{v}+m \in \tilde{V}_0^\phi$, the left-hand side and the right-hand side are both in $\mathrm{Im}(\phi)$. As $\mathrm{Im}(\phi)$ is a (convex) interval, it follows that $m^*_{\tilde{v}}+m \in \mathrm{Im}(\phi)$. Hence, by (I.4), $\tilde{v}+m \sim^* m^*_{\tilde{v}}+m$. But this implies that

$$U(\tilde{v}+m) = m^*_{\tilde{v}} + m = U(\tilde{v}) + m,$$

as desired.

Next, let us show property (v). Assume $r \in V_0^\phi$ and $\tilde{v}, \tilde{v}+r \in \tilde{V}_0^\phi$ with $\tilde{v}, r$ being pc. Because $v \sim^* m^*_{\tilde{v}}$ and $r \sim^* m^*_r$, we have

$$m^*_{\tilde{v}} + m^*_r \leq \max_{w,s} \tilde{v}^w(s) + \max_s r(s) = \max_w \max_s (\tilde{v}^w(s) + r(s)) \in \mathrm{Im}(\phi),$$

where the last equality holds by the assumed comonotonicity of $\tilde{v}$ and r. The right-hand side is in $\mathrm{Im}(\phi)$ because we have assumed that $\tilde{v}+r$ only takes values in $\mathrm{Im}(\phi)$. Similarly one can show that $m^*_{\tilde{v}} + m^*_r \geq \min_w \min_s (\tilde{v}^w(s) + r(s)) \in \mathrm{Im}(\phi)$ and therefore we obtain as before that $m^*_{\tilde{v}} + m^*_r \in \mathrm{Im}(\phi)$. Hence, it follows from A7 that $\tilde{v}+r \sim^* m^*_{\tilde{v}}+m^*_r$. Thus,

$$U(\tilde{v}+r) = m^*_{\tilde{v}} + m^*_r = U(\tilde{v}) + U(r).$$

Hence, U satisfies (v).

To prove property (vi), let $v \in V_0^\phi$ and note that v, v is pc. Thus, $U(2v) = U(v+v) = 2U(v)$, by property (v) whenever $2v \in V_0^\phi$. Upon iterating this argument, $U(\lambda v) = \lambda U(v)$ for all rational non-negative λ such that $\lambda v \in V_0^\phi$. By A3, U is continuous on V_0^ϕ. Hence, $U(\lambda v) = \lambda U(v)$ for all $\lambda \geq 0$ such that $\lambda v \in V_0^\phi$.

Property (viii) follows from Lemma 25 in Maccheroni, Marinacci and Rustichini (2006).

Step 2:

We prove that there exists a non-constant, non-decreasing and continuous function $\psi : [0,1] \to [0,1]$ with $\psi(0) = 0$ and $\psi(1) = 1$ such that, for all $v \in V_0^\phi$,

$$U(v) = \int_{-\infty}^{0} (\psi(1 - F_v(t)) - 1)\,\mathrm{d}t + \int_{0}^{\infty} \psi(1 - F_v(t))\,\mathrm{d}t. \tag{I.5}$$

Denote by $V_0^{[0,1]}$ all random variables in V_0 that only take values between 0 and 1. As $\phi(0) = 0$ and $\phi(1) = 1$, clearly $V_0^{[0,1]}$ is a subset of V_0^ϕ. Denote $\tilde{V}_0^{[0,1]} \subset \tilde{V}_0^\phi$ similarly. The proof of Step 2 then consists of the following two parts:

(I) First, we show that it is sufficient to prove (I.5) for $v \in V_0^{[0,1]}$.

(II) Then, we show that our Axiom A7, when restricted to V_0, corresponds to Axiom A7D used by Yaari (1987). Thus, we conclude that (I.5) holds.

Part (I): It is sufficient to prove (I.5) for $v \in V_0^{[0,1]} \subset V_0^{\phi}$ because, for any $v \in V_0^{\phi}$, there exists a and $1 \geq b > 0$ such that $0 \leq a + bv \leq 1$ (where the inequalities hold for all outcomes). Hence, by (vi) and (vii),

$$U(v) = \frac{1}{b}U(a+bv) - \frac{a}{b} = \frac{1}{b}\int (a+bv)\,\mathrm{d}\nu_{\psi} - \frac{a}{b} = \int v\,\mathrm{d}\nu_{\psi}.$$

Therefore, it is indeed sufficient to prove (I.5) for $v \in V_0^{[0,1]}$.

Part (II): We need the following lemma:

Lemma I.5 *Maintain Axioms A1-A6 (on $\tilde{V}_0^{\phi}$). On the space V_0^{ϕ}, Axiom A7D is then implied by A7, i.e., for $v, u, r \in V_0^{\phi}$ with v, r and u, r pc and $v + r, u + r \in V_0^{\phi}$, we have*

$$\text{for every } \alpha \in (0,1), v \succeq^* u \Rightarrow \alpha v + (1-\alpha)r \succeq^* \alpha u + (1-\alpha)r. \tag{I.6}$$

Proof. Suppose that $v \succeq^* u$ and that the implication

$$v \succeq^* u \Rightarrow v + r' \succeq^* u + r' \tag{I.7}$$

holds for pc v, r' and u, r'. Let $\alpha \in (0,1)$. Since $0 \in \mathrm{Im}(\phi)$, we have that $v, u, r \in \mathrm{Im}(\phi)$ implies that also $\alpha v, \alpha u, (1-\alpha)r \in \mathrm{Im}(\phi)$.

Note that $\alpha v + (1-\alpha)r \succeq^* \alpha u + (1-\alpha)r$ would then follow directly from (I.7) if we could show that $\alpha v \succeq^* \alpha u$ since then,

$$\alpha v + (1-\alpha)r = \alpha v + r' \succeq^* \alpha u + r' = \alpha u + (1-\alpha)r,$$

with $r' := (1-\alpha)r$. But $\alpha v \succeq^* \alpha u$ is an immediate consequence of the fact that U is a numerical representation of $\succeq^*$ on V_0^{ϕ}, and satisfies (vi) of Step 1 above. □

Next, as in Yaari (1987), we shall refer to $1 - F(t)$, with $F(t)$ a CDF, as a decumulative distribution function (DDF). We suppose in the remainder of this proof that F is supported on the unit interval. The (generalized) inverse of a DDF is a reflected (in $t = 1/2$) quantile function, $F^{-1}(1-t)$. In view of the neutrality axiom (Axiom A2), $\succeq^*$ also induces a preference relation on the space of conditional reflected quantile functions (inverse DDFs): given $\tilde{v} \in \tilde{V}_0^{[0,1]}$ with conditional quantile function $q_{\tilde{v}}$, we can define its conditional reflected quantile function $\tilde{G}_{\tilde{v}}$ by

$$\tilde{G}_{\tilde{v}}(.,t) = q_{\tilde{v}}(.,1-t), \qquad t \in [0,1]. \tag{I.8}$$

Now define

$$\tilde{G}_{\tilde{v}}(\succeq^*)\tilde{G}_{\tilde{u}} \qquad \text{if, and only if,} \qquad \tilde{v} \succeq^* \tilde{u}.$$

With this definition, we have defined a preference relation, $(\succeq^*)$, on the convex space

$$\begin{aligned}\tilde{\Gamma} = \{\tilde{G} :\ & W \times [0,1] \to [0,1] \mid \text{For every fixed } t \in [0,1], \\ & \tilde{G}(.,t) \text{ is } \Sigma'\text{-measurable. For every fixed } w \in W, \\ & \tilde{G}(w,.) \text{ is a decreasing and right-continuous step function with} \\ & \tilde{G}(w,1) = 0\}. \end{aligned} \tag{I.9}$$

Indeed, every conditional reflected quantile function is in $\tilde{\Gamma}$ and for every element $G \in \tilde{\Gamma}$, there exists a random variable $\tilde{v} \in \tilde{V}_0^{[0,1]}$ such that $G = \tilde{G}_{\tilde{v}}$.[29] For simplicity, we will henceforth denote the preference relations on the spaces $\tilde{\Gamma}$ and $\tilde{V}_0^{[0,1]} \subset \tilde{V}_0^{\phi}$ both by $\succeq^*$, too. We define Γ as the subspace of all elements in $\tilde{\Gamma}$ that carry no ambiguity, i.e.,

$$\Gamma = \{G \in \tilde{\Gamma} \mid \text{for all } w_1, w_2 \in W :\ \ G(w_1,\cdot) = G(w_2,\cdot)\}.$$

Lemma I.5 implies that, on the space of non-negative random variables in V_0 bounded by one, Axioms A1-A4 and A7D hold. Similar to Yaari (1987), by neutrality (A2), this induces a preference relation on the space of conditional reflected quantile functions, Γ, simply denoted by $\succeq^*$, that satisfies weak and non-degenerate order, continuity, certainty first-order stochastic dominance and the independence axiom.[30] Therefore, by the mixture space theorem (Herstein and Milnor, 1953), there exists a non-constant, non-decreasing and continuous function $\psi : [0,1] \to [0,1]$ such that the corresponding

[29] The latter statement can be verified as follows. Recall that if $U^{\cdot}$ is uniformly distributed on the unit interval and q_X is the quantile function of the random variable X, then $q_X(U^{\cdot})$ has the same distribution as X. Thus, if we define the random variable $\tilde{v} \in \tilde{V}_0^{[0,1]}$ through $\tilde{v}_{\tilde{G}} = \tilde{G}(w, 1-U^w)$, then the conditional reflected quantile function of $\tilde{v}$ is equal to $\tilde{G}$. Furthermore, by neutrality, $\tilde{G}_1(\succeq^*)\tilde{G}_2$ if, and only if, $\tilde{v}_{\tilde{G}_1} \succeq^* \tilde{v}_{\tilde{G}_2}$.

[30] The independence axiom asserts that if, for DDFs $G_1, G_2, G_3 \in \Gamma$, $G_1 \succeq^* G_2$, then, for every $\alpha \in (0,1)$, $\alpha G_1 + (1-\alpha)G_3 \succeq^* \alpha G_2 + (1-\alpha)G_3$.

numerical representation is given by

$$U(v) = -\int_0^1 \psi(t) G_v(., \mathrm{d}t) = \int_0^1 \psi(G_v^{-1}(., t))\, \mathrm{d}t = \int_0^1 \psi(1 - F_v(., t))\, \mathrm{d}t,$$

where G_v is defined by (I.8). Hence, (I.5) holds. Finally, it is straightforward to verify that $U(m) = m$ for all $m \in [0, 1]$ (see Step 1 above) implies that we must have $\psi(0) = 0$ and $\psi(1) = 1$.

Steps 3+4:

Recall Step 1. By construction, $U(0) = 0$. As U satisfies (i)-(viii), U may be identified with a concave and normalized niveloid[31] on the space of bounded, Σ'-measurable functions on W; see Lemma 25 in Maccheroni, Marinacci and Rustichini (2006). Duality results in convex analysis for niveloids (see Lemma 26 in Maccheroni, Marinacci and Rustichini, 2006) then yield that, for all $v' \in V_0^{\prime,\phi}$,

$$U(v') = \min_{Q \in \Delta(W, \Sigma')} \left\{ \mathbb{E}_Q \left[v' \right] + c_{\min}(Q) \right\}, \tag{I.10}$$

with $c_{\min}$ defined by

$$c_{\min}(Q) = \sup_{v' \in V_0^{\prime,\phi}} \left\{ U(v') - \mathbb{E}_Q \left[v' \right] \right\} \geq U(0) = 0, \tag{I.11}$$

and being the unique minimal function satisfying (I.10). As $U(m) = m$ for all $m \in \mathrm{Im}(\phi)$, there exists a Q such that $c(Q) < \infty$. Now we have

$$0 = U(0) = \min_{Q \in \Delta(W, \Sigma')} c(Q).$$

In particular, c is grounded, convex and lower-semicontinuous.

For $\tilde{v} \in \tilde{V}_0^{\phi}$, define $m^*_{\tilde{v}^w} \in \mathrm{Im}(\phi)$ as the corresponding *certainty equivalent* of $\tilde{v}$ in the state of the world w, i.e.,

$$m^*_{\tilde{v}^w} = U(\tilde{v}^w) = \int \tilde{v}^w \, \mathrm{d}\nu_\psi;$$

see Lemma I.4. Set $\bar{v}^w = m^*_{\tilde{v}^w}$. Clearly, $\bar{v}$ is independent of s. Furthermore, by the Theorem of Tornelli, $\bar{v}$ is Σ'-measurable. In particular, $\bar{v}$ is in $V_0^{\prime,\phi}$. Observe that, for every $w \in W$, $U(\bar{v}^w) = U(m^*_{\tilde{v}^w}) = U(U(\tilde{v}^w)) = U(\tilde{v}^w)$, where we have used in the last

[31] The mapping U from $V_0^{\prime,\phi}$ to $\mathrm{Im}(\phi)$ is a concave and normalized niveloid if it is concave, Lipschitz continuous with respect to the $||.||_\infty$-norm, and satisfies $U(m) = m$ for all $m \in \mathrm{Im}(\phi)$.

equality that, for all $m \in \mathrm{Im}(\phi)$, $U(m) = m$ and by the monotonicity property (iv) of Step 1,

$$\begin{aligned}\mathrm{Im}(\phi) \ni \min_s \tilde{v}^w(s) = U(\min_s \tilde{v}^w(s)) \leq U(\tilde{v}^w)\\ \leq U(\max_s \tilde{v}^w(s)) = \max_s \tilde{v}^w(s) \in \mathrm{Im}(\phi),\end{aligned}$$

so that $U(\tilde{v}^w) \in \mathrm{Im}(\phi)$. Hence, property (iv) of Step 1 implies that $U(\bar{v}) = U(\tilde{v})$. This entails that, for all $\tilde{v} \in \tilde{V}_0^\phi$,

$$\begin{aligned}U(\tilde{v}) = U(\bar{v}) &= \min_{Q\in\Delta(W,\Sigma')} \{\mathbb{E}_Q[\bar{v}] + c(Q)\}\\ &= \min_{Q\in\Delta(W,\Sigma')} \left\{\int U(m^*_{\tilde{v}^w}) Q(\mathrm{d}w) + c(Q)\right\}\\ &= \min_{Q\in\Delta(W,\Sigma')} \left\{\int U(\tilde{v}^w) Q(\mathrm{d}w) + c(Q)\right\}\\ &= \min_{Q\in\Delta(W,\Sigma')} \left\{\int \left(\int_{-\infty}^0 (\psi(1 - F_{\tilde{v}^w}(t)) - 1)\,\mathrm{d}t\right.\right.\\ &\qquad\left.\left. + \int_0^\infty \psi(1 - F_{\tilde{v}^w}(t))\,\mathrm{d}t\right) Q(\mathrm{d}w) + c(Q)\right\}\\ &= \min_{Q\in\Delta(W,\Sigma')} \left\{\mathbb{E}_Q\left[\int \tilde{v}^\cdot\,\mathrm{d}\nu_\psi\right] + c(Q)\right\},\end{aligned}$$

where we have used (I.10) in the second and (I.5) in the fifth equalities. This proves the "only if" part of Theorem I.2(α).

The proof of Theorem I.2(β) now follows by defining, for all $\tilde{v}, \tilde{u} \in \tilde{V}^\phi$,

$$\begin{aligned}\tilde{v} \succeq^* \tilde{u} \Leftrightarrow U(\tilde{v}) := &\min_{Q\in\Delta(W,\Sigma')} \left\{\mathbb{E}_Q\left[\int \tilde{v}^\cdot\,\mathrm{d}\nu_\psi\right] + c(Q)\right\}\\ \geq &\min_{Q\in\Delta(W,\Sigma')} \left\{\mathbb{E}_Q\left[\int \tilde{u}^\cdot\,\mathrm{d}\nu_\psi\right] + c(Q)\right\} = U(\tilde{u}).\end{aligned}$$

The extension is unique, as it may be seen as before that it follows from our axioms that the functional U defined above is concave on $\tilde{V}^\phi$. Therefore, U is continuous on the interior of its domain, which gives a unique extension from the dense subspace $\tilde{V}_0^\phi$ to $\tilde{V}^\phi$. □

Proof of Theorem 3.6. For the proof of Theorem 3.6 we need the following lemma:

Lemma I.6 *A7^0 with "$\oplus$" replaced by "$+$" for random variables only taking values in Im(ϕ) (i.e., with $\tilde{V}_0, V_0$ replaced by $\tilde{V}_0^\phi, V_0^\phi$) implies that, for $\tilde{v}, \tilde{u} \in \tilde{V}_0^\phi$, $\tilde{v} \succeq^* \tilde{u}$ if, and* *only if,* $\lambda\tilde{v} \succeq^* \lambda\tilde{u}$ *for every* $\lambda \geq 0$ *such that* $\lambda\tilde{v}, \lambda\tilde{u} \in \tilde{V}_0^\phi$.

Proof. The proof of the "if" part is straightforward. Let us prove the "only if" part. So, suppose that $\tilde{v} \succeq^* \tilde{u}$. If $\lambda \in [0,1]$, then $\lambda\tilde{v} \succeq^* \lambda\tilde{u}$ follows directly from Axiom A7^0 with $\alpha = \lambda$ and $r = 0$. If $\lambda > 1$, then let us suppose that $\lambda\tilde{u} \succ^* \lambda\tilde{v}$ would hold. Defining $\alpha = \frac{1}{\lambda} \in (0,1)$ yields, by A7^0,

$$\tilde{u} = \alpha\lambda\tilde{u} + (1-\alpha)0 \succ^* \alpha\lambda\tilde{v} + (1-\alpha)0 = \tilde{v},$$

which is a contradiction. Hence, indeed $\lambda\tilde{v} \succeq^* \lambda\tilde{u}$ for every $\lambda \geq 0$. □

Thus, Axiom A7^0 implies that the preference relation is scale invariant on $\tilde{V}_0^\phi$, whereas Axiom A7 only implies scale invariance on V_0^ϕ (formally, via Lemma I.5). The next proposition shows explicitly that Axiom A7^0 is stronger than Axiom A7:

Proposition I.7 *Axiom A7^0 implies Axiom A7, with "$\oplus$" replaced by "$+$" for random variables only taking values in Im(ϕ) (i.e., with $\tilde{V}_0$, V_0 replaced by $\tilde{V}_0^\phi$, V_0^ϕ).*

Proof. First, in view of Lemma I.6, we can extend $\succeq^*$ consistently to $\tilde{V}_0$ by defining, for *any* $\tilde{v}, \tilde{u} \in \tilde{V}_0$, that $\tilde{v} \succeq^* \tilde{u}$ if

$$\frac{\min(-\inf\phi, 1)}{2\max(|\tilde{v}| \vee |\tilde{u}|)}\tilde{v} \succeq^* \frac{\min(-\inf\phi, 1)}{2\max(|\tilde{v}| \vee |\tilde{u}|)}\tilde{u},$$

where in case both random variables are degenerate at zero the scaling factor should be set equal to 1. Note that as $0, 1 \in \text{Im}(\phi)$ and $\text{Im}(\phi)$ is open, we have $\inf\phi < 0$, and therefore $\inf\phi < \frac{\min(-\inf\phi,1)}{2\max(|\tilde{v}|\vee|\tilde{u}|)}\tilde{v} < 1$ and a similar inequality holds for the scaled $\tilde{u}$, such that the scaled random variables only take values in $\text{Im}(\phi)$. Clearly, the (scaled) extension of $\succeq^*$ to $\tilde{V}_0$ is again scale invariant in the sense that, *mutatis mutandis*, Lemma I.6 holds on the entire space $\tilde{V}_0$. Thus, Axiom A7^0 also holds on the entire space $\tilde{V}_0$.

Next, let us show the proposition. Suppose that $\tilde{v} \succeq^* \tilde{u}$ and that $\tilde{v}, \tilde{u}$ and r are pc. Let $\alpha \in (0,1)$. Then, by Lemma I.6 and the paragraph above, under Axiom A7^0, $\frac{1}{\alpha}\tilde{v} \succeq^* \frac{1}{\alpha}\tilde{u}$. Next, let $\bar{r} = \frac{r}{1-\alpha}$. Then, we obtain from Axiom A7^0 that

$$\tilde{v} + r = \alpha\left(\frac{1}{\alpha}\tilde{v}\right) + (1-\alpha)\bar{r} \succeq^* \alpha\left(\frac{1}{\alpha}\tilde{u}\right) + (1-\alpha)\bar{r} = \tilde{u} + r.$$

Hence, A7 is indeed satisfied. □

Because Axioms A1-A6 and A7^0 imply A1-A7, we may, as before, obtain a numerical representation U for $\succeq^*$. Noticing that, by Lemma I.6, U must be positively homogeneous, we can as in the proof of Proposition I.7 extend U to the entire space $\tilde{V}_0$, and it follows from classical results in convex analysis that the minimal function $c_{\min}$ in (I.10) can then be chosen to only take the values zero or infinity.

The unique extension of $\succeq^*$ to $\tilde{V}$ can be seen as in the proof of Theorem I.2. The proof of Theorem 3.6(b) follows from Föllmer and Schied (2016), Chapter 4. This completes the proof of Theorem 3.6. □

Proof of Proposition 4.1. Recall that $v \succeq^* w$ if, and only if, $\phi^{-1}(v) \succeq \phi^{-1}(w)$, with ϕ from Proposition 3.1. Furthermore, recall the equivalence asserted by Lemma I.1. Now one may verify that the corresponding Axioms A1-A3 are already sufficient to guarantee the existence of a numerical representation of $\succeq^*$. As in the proof of Theorem 3.5, denote it by U. By (I.3), U may be seen to be concave and upper-semicontinuous on the space $V_0^{\prime,\phi}$. Therefore, Theorem 4.1 in Dana (2005) implies that $\succeq^*$ respecting SSD is equivalent to U being law invariant under P'. Using the definition of $\succeq^*$ then shows the first part of the proposition. Noting that, if ϕ is concave, v' dominates u' in SSD if, and only if, $\phi(u')$ dominates $\phi(v')$ in SSD finishes the proof of Proposition 4.1. □

Proof of Proposition 4.2. If $\succeq^*$ is more ambiguity averse than $\succeq$, then $\succeq$ and $\succeq^*$ agree on V_0, and therefore we may choose a positive affine transformation such that $\phi^* = \phi$, and furthermore $\psi^* = \psi$. This implies that

$$c^*_{\min}(Q) = \sup_{v' \in V_0'} \left\{ m^*_{\phi(v')} - \mathbb{E}_Q[\phi(v')] \right\} \geq \sup_{v' \in V_0'} \left\{ m_{\phi(v')} - \mathbb{E}_Q[\phi(v')] \right\} = c_{\min}(Q).$$

This proves the "only if" part. To prove the "if" part, suppose that $c^* \geq c$, $\psi^* = \psi$ and $\phi^* = \phi$. Then $\tilde{v} \succeq v$, with $\tilde{v} \in \tilde{V}_0$ and $v \in V_0$, entails that

$$\begin{aligned} m^*_v = m_v &\leq \phi^{-1}\Big(\min_{Q \in \Delta(W,\Sigma')} \left\{ \mathbb{E}_Q \left[\int \phi(\tilde{v}^{\cdot}) \, \mathrm{d}\nu_{\psi} \right] + c_{\min}(Q) \right\} \Big) \\ &\leq \phi^{*,-1}\Big(\min_{Q \in \Delta(W,\Sigma')} \left\{ \mathbb{E}_Q \left[\int \phi^*(\tilde{v}^{\cdot}) \, \mathrm{d}\nu_{\psi^*} \right] + c^*_{\min}(Q) \right\} \Big) = m^*_{\tilde{v}}. \end{aligned}$$

Hence, $\tilde{v} \succeq^* v$. □

Proof of Proposition 4.3. Assume Axioms A1-A8. By Theorem 3.5, there exist functions ϕ, ψ and c such that (3.6) holds. Set $P' = \arg\min_Q c_{\min}(Q)$. Because $c_{\min}$ is grounded, $c_{\min}(P') = 0$. Denote by $\succeq^{\mathrm{AN}}$ the ambiguity neutral agent with measure P', utility function ϕ and probability weighting function ψ. Suppose that $\tilde{v} \succeq v$, $\tilde{v} \in \tilde{V}_0$ and

$v \in V_0$. Then,

$$m_v \leq \phi^{-1}\Big(\min_{Q\in\Delta(W,\Sigma')}\Big\{\mathbb{E}_Q\Big[\int \phi(\tilde{v}^{\cdot})\,\mathrm{d}\nu_\psi\Big] + c_{\min}(Q)\Big\}\Big)$$
$$\leq \phi^{-1}\Big(\mathbb{E}_{P'}\Big[\int \phi(\tilde{v}^{\cdot})\,\mathrm{d}\nu_\psi\Big] + c_{\min}(P')\Big) = \phi^{-1}\Big(\mathbb{E}_{P'}\Big[\int \phi(\tilde{v}^{\cdot})\,\mathrm{d}\nu_\psi\Big]\Big).$$

□

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