Geometric BSDEs

Roger J. A. Laeven*
Dept. of Quantitative Economics
University of Amsterdam, CentER
and EURANDOM, The Netherlands
R.J.A.Laeven@uva.nl

Emanuela Rosazza Gianin
Dept. of Statistics and Quantitative Methods
University of Milano-Bicocca, Italy
emanuela.rosazza1@unimib.it

Marco Zullino
Dept. of Mathematics and Applications
University of Milano-Bicocca, Italy
M.Zullino@campus.unimib.it

This Version: September 9, 2025

Abstract

We introduce and develop the concepts of Geometric Backward Stochastic Differential Equations (GBSDEs, for short) and two-driver BSDEs. We demonstrate their natural suitability for modeling continuous-time dynamic return risk measures. We characterize a broad spectrum of associated, auxiliary ordinary BSDEs with drivers exhibiting growth rates involving terms of the form $y|\ln(y)|+|z|^2/y$. We establish the existence, regularity, uniqueness, and stability of solutions to this rich class of ordinary BSDEs, considering both bounded and unbounded coefficients and terminal conditions. We exploit these results to obtain corresponding results for the original two-driver BSDEs. Finally, we apply our findings within a GBSDE framework for representing the dynamics of return and star-shaped risk measures including (robust) L^p-norms, and analyze functional properties.

Keywords: Geometric BSDEs; Two-driver BSDEs; Logarithmic non-linearity and singularity at zero; Dynamic return and star-shaped risk measures; Dynamics of L^p-norms.

MSC 2020 Classification: *Primary*: 60H10, 60H30; *Secondary*: 91B06, 91B30, 62P05.

OR/MS Classification: Probability: Diffusion; Probability: Stochastic model applications; Decision analysis: Risk.

1 Introduction

The geometric nature of random growth and risky asset price processes in continuous time is well recognized and omnipresent in stochastic analysis and its wide variety of applications, with the geometric Brownian motion (GBM, for short) as the canonical elementary example. By contrast, continuous-time robust dynamic risk assessment via backward stochastic differential equations (BSDEs) usually occurs in an arithmetic environment (see, e.g., [26, 32, 34, 37, 41, 43, 45, 48]).

In the recent literature (see, e.g., [9, 10, 38, 39] and also [1, 8, 23, 31]), a growing interest has been devoted to return risk measures that assess relative financial positions (or log returns) instead of absolute positions as with monetary risk measures ([18, 30]). This relative evaluation, reminiscent of the classical notion of relative risk aversion ([50]),[1] naturally leads to a multiplicative, i.e., geometric, structure for return risk measures and their acceptance sets. These considerations — together with the fact that a wide family of dynamic monetary risk measures is induced by BSDEs (see [11, 17, 19, 49, 52]) — motivate the study of geometric forms of

*Corresponding author.

[1] That is, monetary and return risk measures exhibit a similar relationship as absolute and relative risk aversion, arithmetic and geometric means, arithmetic and geometric growth, and arithmetic and geometric Brownian motion.

BSDEs, both from a purely mathematical point of view as well as for applications. To our best knowledge, the only paper dealing with continuous-time dynamic return risk measures is the recent [40]. There, the main aim is to characterize the properties of dynamic return and star-shaped risk measures induced via BSDEs that satisfy L^∞- or L^2-standard assumptions. The authors in [40] did not attempt to describe the geometric nature of dynamic return risk measures, analyze inherent properties such as multiplicative (i.e., geometric) convexity, or represent dynamic (robust) L^p-norms. To address these problems, we develop in this work a novel approach, by changing the stochastic differential equations that drive the dynamics of return risk measures, and establishing their existence, regularity, uniqueness, and stability.

More specifically, we introduce and analyze a novel class of BSDEs, which we term *geometric BSDEs*. Just like the geometric mean is connected to the arithmetic mean via $\mathbb{G}[X] := \exp(\mathbb{E}[\ln(X)])$, geometric BSDEs naturally arise from the substitution $\tilde{\rho}_t(X) := \exp(\rho_t(\ln(X)))$ for any $t \in [0,T]$, where $T > 0$ is a fixed finite time horizon and $Y_\cdot := \rho_\cdot(\ln(X))$ is the first component of the solution $(Y_\cdot, Z_\cdot)$ to the following quadratic BSDE:

$$Y_t = \ln(X) + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s.$$

Here, $(W_t)_{t\in[0,T]}$ is a standard n-dimensional Brownian motion, the driver $f : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is $d\mathbb{P} \times dt$-a.s. continuous in (y,z) and satisfies the growth condition $|f(t,y,z)| \leq \alpha_t + \beta_t|y| + \gamma_t|z| + \delta|z|^2$ for suitable positive stochastic coefficients α, β, γ, and a constant $\delta > 0$, while X is a strictly positive random variable (i.e., $\mathbb{P}(X > 0) = 1$) verifying some further integrability conditions. Under appropriate hypotheses, the dynamics of $\tilde{Y}_\cdot := \tilde{\rho}_\cdot(X)$ can be obtained by Itô's formula and can be represented as the first component of the solution to a geometric BSDE (GBSDE, for short) of the form

$$\tilde{Y}_t = X + \int_t^T \tilde{Y}_s \tilde{f}(s, \tilde{Y}_s, \tilde{Z}_s)ds - \int_t^T \tilde{Y}_s \tilde{Z}_s dW_s, \tag{1.1}$$

whose multiplicative nature is even more evident when cast in its infinitesimal version:

$$\begin{cases} -\frac{d\tilde{Y}_t}{\tilde{Y}_t} = \tilde{f}(t, \tilde{Y}_t, \tilde{Z}_t)dt - \tilde{Z}_t dW_t, \\ \tilde{Y}_T = X. \end{cases}$$

Here, the driver $\tilde{f} : \Omega \times [0,T] \times \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ is defined by the formula $\tilde{f}(t,y,z) := f(t, \ln(y), z) - \frac{1}{2}|z|^2$, thus $\tilde{f}$ verifies the growth rate

$$|\tilde{f}(t,y,z)| \leq \alpha_t + \beta_t|\ln(y)| + \gamma_t|z| + \left(\delta + \frac{1}{2}\right)|z|^2. \tag{1.2}$$

As will become apparent, GBSDEs generalize the *geometric* martingale representation theorem ([7, 21]), which corresponds to $\tilde{f} \equiv 0$, and are naturally connected to monotone and positively homogeneous dynamic risk measures (a.k.a. dynamic return risk measures; see [9, 10, 38, 40]).

To provide a general, systematic analysis of the existence, regularity, uniqueness and stability of solutions, we embed the GBSDE (1.1) by deploying the broader concept of *two-driver BSDEs*. Specifically, we consider two drivers, $g_1 : \Omega \times [0,T] \times \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ and $g_2 : \Omega \times [0,T] \times \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$, leading to the equation

$$Y_t = X + \int_t^T g_1(s, Y_s, Z_s)ds - \int_t^T g_2(s, Y_s, Z_s)dW_s. \tag{1.3}$$

Two-driver BSDEs are discussed in the seminal paper [47]. However, these authors imposed restrictive hypotheses on g_2, requiring a bi-Lipschitz condition in (y,z) and injectivity with respect to z, while g_1 was assumed to be Lipschitz in (y,z). Whereas significant progress has been made to weaken the hypothesis of Lipschitzianity for g_1 (see, among many others, [3, 4, 12, 22, 36]), little attention has been given to BSDEs with general double drivers, as the choice $g_2(t, y, z) = z$ has been convenient in many applications, particularly in describing the dynamics of monetary risk measures (e.g., [7, 11]). Nevertheless, the geometric structure evident in Equation (1.1) yields $g_2(t, y, z) = yz$ and it is clear that the bi-Lipschitz assumption fails for such g_2. Therefore, in our framework, we relax the bi-Lipschitz condition by only requiring that the $\mathbb{R}^n$-norm of g_2 grows 'sufficiently fast' in (y, z). Our general assumptions ensure that Equation (1.1) can be obtained as a specific case of Equation (1.3), and thus our results can be utilized to explore the properties of dynamic return risk measures induced through GBSDEs.

Our results concerning two-driver BSDEs are in substantial part — but importantly not fully — obtained by deriving corresponding results for an auxiliary class of ordinary BSDEs, revealing an interesting connection between these two families of BSDEs. The examination of this class of ordinary BSDEs is motivated by setting $\bar{Z}_t := \tilde{\rho}_t \tilde{Z}_t$ in Equation (1.1), yielding an ordinary BSDE of the following form:

$$\bar{Y}_t = X + \int_t^T \bar{Y}_s \bar{f}(s, \bar{Y}_s, \bar{Z}_s)ds - \int_t^T \bar{Z}_s dW_s,$$

where $\bar{f} : \Omega \times [0, T] \times \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ is defined as $\bar{f}(t, y, z) := \tilde{f}(s, y, z/y)$, assuming $\tilde{f} \geq 0$, and Equation (1.2) induces the growth rate $\bar{f}(t, y, z) \leq \alpha_t + \beta_t |\ln(y)| + \gamma_t |z| + \delta |z|^2/y$. By analogy and under suitable hypotheses, Equation (1.3) can be transformed into an ordinary BSDE whose driver $g : \Omega \times [0, T] \times \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ verifies the following growth rate:

$$g(t, y, z) \leq \alpha_t y + \beta_t y |\ln(y)| + \gamma_t |z| + \delta |z|^2/y. \tag{1.4}$$

In other words, an auxiliary (but not sufficient) tool to study the properties of the original two-driver BSDE in Equation (1.3) is provided by the ordinary BSDE

$$Y_t = X + \int_t^T g(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s, \tag{1.5}$$

where g satisfies the general condition (1.4), exhibiting a logarithmic non-linearity and singularity at zero. The introduction and comprehensive analysis of general two-driver BSDEs, encompassing GBSDEs, and their precise connections to dynamic risk measures, constitutes our main methodological contribution.

We establish general existence, regularity, uniqueness and stability results for the two-driver BSDE (1.3), exploiting (1.4)–(1.5), considering both bounded and unbounded stochastic coefficients α, β, γ and terminal condition X. These results are mathematically involved; in the subsequent paragraphs, we highlight the key mathematical challenges, and the proof strategies we introduce, in comparison to the existing literature. While existence, uniqueness and stability results for ordinary BSDEs with a $y|\ln(y)|$-growth rate have been established in [5] and existence and uniqueness for a $|z|^2/y$-growth rate have been analyzed in [6] (without considering stability of the solution), to the best of our knowledge, this is the first contribution that simultaneously incorporates both the logarithmic non-linearity $y|\ln(y)|$ and the singularity at zero represented by $|z|^2/y$ into the general form (1.4). Furthermore, our results are based on different assumptions compared to those in [5] and relax the assumptions in [6]. Moreover, our results ultimately apply to two-driver BSDEs of the form (1.3).

The proof techniques to establish the existence of the solution to (1.4)–(1.5) draw some inspiration from the insightful analyses in [2, 4]. Demonstrating the regularity of the z-component of the solution necessitates a novel approach that is based on *a priori* estimates. Settings to achieve square-integrability of the z-component for a driver exhibiting a $|z|^2/y$-growth rate were previously unexplored except for specific cases, such as when X is bounded and bounded away from zero ([6]). We achieve this integrability without imposing additional hypotheses, while also incorporating the logarithmic non-linearity. The regularity we obtain for the (y, z)-solution pair is optimal, recovering existing special cases. Additionally, we develop further regularities under bounded coefficients α, β, γ and terminal condition X.

To establish uniqueness, we assume g to be jointly convex in (y, z). Initially proposed in [13] and further explored in [20, 29], the convexity assumption enables us to derive a comparison theorem, yielding uniqueness. While our approach builds on the idea in [13] of estimating the so-called θ-difference, $Y - \theta Y'$, between two solutions to obtain comparison results, our proofs faced significant challenges due to the $y|\ln(y)| + |z|^2/y$-growth rate terms. Unlike [13], which assumes the existence of exponential moments of order $p \geq 1$ for the terminal condition X, we do not require the existence of p-moments of all orders, which can be regarded as the natural geometric counterpart of exponential moments. In this sense, our results are linked to the sharper uniqueness results obtained in [20], but we achieve this using a completely different proof strategy (not based on a stochastic control problem). Furthermore, these earlier works assume convexity of g with respect to z and a monotonicity condition in y, to eliminate dependence of g on the state variable. However, we cannot rely on this kind of monotonicity due to the logarithmic non-linearity. The hypothesis of joint convexity in (y, z) proves to be suitable to address the logarithmic non-linearity, as it allows us to utilize a stochastic Bihari inequality — a stochastic generalization of a Grönwall lemma — recently introduced in [33]. Uniqueness of the solution for a $|z|^2/y$-growth rate with unbounded terminal condition is unprecedented in the literature. In [6], the authors establish uniqueness (under convexity of the driver) for a terminal condition X that is bounded and bounded away from zero. We relax both assumptions.

Furthermore, as a byproduct of independent interest, we obtain a stability result for the solution. This result relies on the convexity of the generator and represents the first stability result for a $|z|^2/y$-growth rate, even in the absence of the logarithmic non-linearity. While the proof follows a scheme similar to that of the comparison theorem mentioned above, it requires meticulous verification of the necessary integrability conditions. We note that these stability results can also be employed to refine the stability results presented in [13], as the results allow to handle exponential moments that are not necessarily finite for all $p \geq 1$, dropping the hypothesis of monotonicity in y and assuming joint convexity in (y, z).

As mentioned above, our approach is designed to comprehensively investigate Equation (1.3). That is, exploiting the theory developed for Equation (1.5), we study two-driver BSDEs of the form (1.3). Specifically, we introduce a notion of a solution for two-driver BSDEs, deriving existence of the solution from the analogous results for Equation (1.5). To investigate the regularity of the z-component of the solution to Equation (1.3), we provide a distinct analysis; it *cannot* be derived directly from the single-driver Equation (1.5). In addition, a comparison theorem for two-driver BSDEs is established, yielding also uniqueness of the solution. These results hold under suitable hypotheses on the composition between g_1 and g_2, and substantially generalize the results obtained for GBSDEs. Finally, stability results for the solution to the two-driver BSDE are provided.

Additionally, we present non-trivial findings on dynamic return and star-shaped risk measures induced by GBSDEs. We demonstrate the feasibility of defining dynamic return risk measures (and generalizations thereof) for unbounded terminal conditions and identify sufficient conditions for the drivers that ensure financially meaningful properties, such as multiplicative convexity, positive homogeneity, and star-shapedness. These results are obtained by exploiting the stability properties we have established.

To illustrate our results, we include several examples. In particular, we demonstrate that L^p-norms and robust L^p-norms (i.e., the return counterparts of the monetary entropic and robust entropic risk measures, see [30, 42]) can be described as solutions to GBSDEs. While the relevance of these objects in economics and financial mathematics is well-known, see e.g., [9, 15, 38, 43, 53], to our best knowledge, this is the first attempt to express general dynamic robust L^p-norms as solutions to BSDEs.

The remainder of this paper is structured as follows. In Section 2, we provide the basic framework and review some preliminaries for dynamic return and monetary risk measures. Section 3 introduces the concept of GBSDEs and presents initial results on their existence and uniqueness. Our main results concerning Equation (1.5) are established in Section 4. Section 5 delves into the specifics of the two-driver BSDE in Equation (1.3), also elucidating how these results apply to the realm of GBSDEs. Section 6 demonstrates how GBSDEs can be used to represent dynamic return and star-shaped risk measures and characterizes the properties of the risk measures in terms of those of the GBSDE drivers. Some auxiliary results and all proofs that are not in the main text are collected in Online Appendix A.

2 Preliminaries

In this section, we first introduce the notation and setting utilized in the sequel. Next, we provide the definitions of dynamic return and monetary risk measures.

2.1 Main notation and functional spaces

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Furthermore, let $T > 0$ denote a finite time horizon and let $(W_t)_{t\in[0,T]}$ be a standard n-dimensional Brownian motion defined on $(\Omega, \mathcal{F}, \mathbb{P})$. We equip the probability space with $(\mathcal{F}_t)_{t\in[0,T]}$, the augmented filtration associated to that generated by $(W_t)_{t\in[0,T]}$. We require w.l.o.g. that $\mathcal{F} = \mathcal{F}_T$. All equalities and inequalities between random variables are understood to hold $d\mathbb{P}$-almost surely whereas for stochastic processes they are meant to be valid $d\mathbb{P} \times dt$-a.s., unless specified otherwise. We equip the space L^0 of all $\mathcal{F}$-measurable random variables with the usual pointwise partial order relation, writing $X \geq Y$ when $X(\omega) \geq Y(\omega)$ for $\mathbb{P}$-almost every $\omega \in \Omega$. We use the notation L^0_+ to denote the set of all strictly positive random variables. Given $\mathcal{X} \subseteq L^0$, we set $\mathcal{X}_+ := \mathcal{X} \cap L^0_+$. For each fixed $t \in [0, T]$, given $\mathcal{X} \subseteq L^0$, we denote by $\mathcal{X}(\mathcal{F}_t)$ the space of $\mathcal{F}_t$-measurable random variables belonging to $\mathcal{X}$. For any $p \in [1, +\infty)$, $L^p(\mathcal{F}_T)$ is the set of p-integrable random variables whose norm is denoted by $\|\cdot\|_p$. $L^\infty(\mathcal{F}_t)$ is the space of $\mathcal{F}_t$-measurable and essentially bounded random variables, whose norm is denoted by $\|\cdot\|_\infty$. When no confusion can arise, we write L^p instead of $L^p(\mathcal{F}_T)$, and similarly for the other spaces, without further specification. For any $x, y \in \mathbb{R}^n$ with $n \in \mathbb{N}$, $n > 1$, we write $x \cdot y$ for the usual scalar product in $\mathbb{R}^n$, i.e., $x \cdot y := \sum_{i=1}^n x_i y_i$. For clarity, we sometimes use the notation $x \cdot y$ even when x and y are scalars. We let $\mathbb{R}_+ := \{x \in \mathbb{R} : x > 0\}$.

Next, we define the primary functional spaces that we consider. From now on, we will use "p.p." to denote any predictable process with respect to $(\mathcal{F}_t)_{t\in[0,T]} \otimes \mathcal{B}(0, T)$, with $\mathcal{B}$ the Borel σ-algebra, and valued in $\mathbb{R}^n$, with $n \geq 1$. We let

$$\mathcal{H}^p_T := \left\{ (Y_t)_{t\in[0,T]} \text{ p.p.: } \mathbb{E}\left[\operatorname*{ess\,sup}_{t\in[0,T]} |Y_t|^p \right] < +\infty \right\},$$

$$\mathcal{H}^\infty_T := \left\{ (Y_t)_{t\in[0,T]} \text{ p.p.: } \operatorname*{ess\,sup}_{(t,\omega)\in[0,T]\times\Omega} |Y_t| < +\infty \right\},$$

$$\mathcal{L}^2_T := \left\{ (Y_t)_{t\in[0,T]} \text{ p.p.: } \int_0^T |Y_t|^2 dt < +\infty, \ \mathbb{P}\text{-a.s.} \right\},$$

$$\mathcal{M}^p_T := \left\{ (Y_t)_{t\in[0,T]} \text{ p.p.: } \mathbb{E}\left[\left(\int_0^T |Y_t|^2 dt \right)^{\frac{p}{2}} \right] < +\infty \right\},$$

$$\text{BMO}(\mathbb{P}) := \left\{ (Y_t)_{t\in[0,T]} \text{ p.p.: } \exists C > 0 \text{ s.t. } \mathbb{E}\left[\int_t^T |Y_s|^2 ds \middle| \mathcal{F}_t \right] \leq C \ d\mathbb{P} \times dt\text{-a.s.} \right\}.$$

Sometimes we will employ a different probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$. In that case, we will specify the regularity of a process w.r.t. this measure by writing, e.g., $\mathcal{H}^p_T(\mathbb{Q})$ and analogously for the other spaces. Similarly, we will specify that the expectations are taken w.r.t. $\mathbb{Q}$ by writing $\mathbb{E}_{\mathbb{Q}}[\cdot]$. Henceforth, we use the shorthand notation $X.$ instead of $(X_t)_{t\in[0,T]}$ for a stochastic process. For any $X. \in \mathcal{H}_T^\infty$, we define the norm $\|X\|_\infty^T := \operatorname*{ess\,sup}_{(t,\omega)\in[0,T]\times\Omega} |X_t|$; $\mathcal{H}_T^\infty$ is a Banach space if equipped with $\|\cdot\|_\infty^T$. When dealing with $X. \in \mathrm{BMO}(\mathbb{P})$, we consider the norm $\|X\|_{\mathrm{BMO}} := \sup_{t\in[0,T]} \mathbb{E}[\int_t^T |X_s|^2 ds|\mathcal{F}_t]$. As is well known, $(\mathrm{BMO}(\mathbb{P}), \|\cdot\|_{\mathrm{BMO}})$ forms a Banach space ([35]).

We define the stochastic exponential of $\gamma \in \mathcal{L}_T^2$ as $\mathcal{E}_t^\gamma := \exp\left(\int_0^t \gamma_s dW_s - \frac{1}{2}\int_0^t |\gamma_s|^2 ds\right)$. If $\gamma \in \mathrm{BMO}(\mathbb{P})$, $\mathcal{E}_T^\gamma$ is the density of an equivalent probability measure $\mathbb{Q}^\gamma$ (w.r.t. $\mathbb{P}$) and $W_\cdot^\gamma := W. - \int_0^\cdot \gamma_s ds$ is a $\mathbb{Q}^\gamma$-Brownian motion w.r.t. the same stochastic base of $W.$ (by Girsanov).

In what follows, we also need the space of positive random variables bounded away from zero. Specifically, fixing $\varepsilon > 0$, we introduce the set and respective functional space

$$\mathcal{L}_\varepsilon^\infty(\mathcal{F}_T) := \{Y \in L^\infty(\mathcal{F}_T) : Y \geq \varepsilon \text{ a.s.}\}, \qquad \mathcal{L}^\infty(\mathcal{F}_T) := \bigcup_{\varepsilon>0} \mathcal{L}_\varepsilon^\infty(\mathcal{F}_T).$$

When no confusion is possible, we simply write $\mathcal{L}_\varepsilon^\infty$ and $\mathcal{L}^\infty$, respectively. In addition, when working with a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$, we write $\mathcal{L}^\infty(\mathcal{F}_t)$ (resp. $\mathcal{L}_\varepsilon^\infty(\mathcal{F}_t)$) to indicate the set of random variables in $\mathcal{L}^\infty$ (resp. $\mathcal{L}_\varepsilon^\infty$) that are $\mathcal{F}_t$-measurable. It is important to note that $\mathcal{L}^\infty$ is a subset of L^∞, and this inclusion is strict. We have the following characterization, which will often be used in the subsequent sections:

Lemma 1. *For any $t \in [0,T]$, the space $\mathcal{L}^\infty(\mathcal{F}_t)$ can be identified with the set of random variables given by*

$$L_{\ln}^\infty(\mathcal{F}_t) := \{Y \in L_+^0 : \ \ln(Y) \in L^\infty(\mathcal{F}_t)\}.$$

The proof is a routine verification and is omitted for brevity. We will also make use of the following two related spaces for stochastic processes:

$$\mathcal{L}_{\varepsilon,T}^\infty := \{Y \in \mathcal{H}_T^\infty : \ Y_t \geq \varepsilon \ d\mathbb{P}\times dt\text{-a.s.}\}, \qquad \mathcal{L}_T^\infty := \bigcup_{\varepsilon>0} \mathcal{L}_{\varepsilon,T}^\infty.$$

2.2 Dynamic return and monetary risk measures

In this subsection, we provide definitions of dynamic return and monetary risk measures, used throughout.

Definition 2. *Let $L^\infty(\mathcal{F}_T) \subseteq \mathcal{X} \subseteq L^0(\mathcal{F}_T)$. Then $\rho_t : \mathcal{X}(\mathcal{F}_T) \to \mathcal{X}(\mathcal{F}_t)$ is a risk measure if it is monotone, i.e., for any $X, Y \in \mathcal{X}(\mathcal{F}_T)$ such that $X \geq Y$, it holds that $\rho_t(X) \geq \rho_t(Y)$. A monetary risk measure is a risk measure that additionally verifies cash-additivity, that is, for any $t \in [0,T]$, $X \in \mathcal{X}(\mathcal{F}_T)$ and $\eta_t \in \mathcal{X}(\mathcal{F}_t)$ such that $X + \eta_t \in \mathcal{X}(\mathcal{F}_T)$, it holds that $\rho_t(X + \eta_t) = \rho_t(X) + \eta_t$.*

Let $\mathcal{L}^\infty(\mathcal{F}_T) \subseteq \mathcal{Y} \subseteq L_+^0(\mathcal{F}_T)$. Then $\tilde{\rho}_t : \mathcal{Y}(\mathcal{F}_T) \to \mathcal{Y}(\mathcal{F}_t)$ is a return risk measure if it is monotone on $\mathcal{Y}(\mathcal{F}_T)$ and positively homogeneous with respect to $\mathcal{F}_t$-measurable random variables, that is, for any $t \in [0,T]$, $X \in \mathcal{X}(\mathcal{F}_T)$ and $\xi_t \in \mathcal{L}^\infty(\mathcal{F}_t)$ such that $\xi_t \cdot X \in \mathcal{Y}(\mathcal{F}_T)$, it holds that $\tilde{\rho}_t(\xi_t \cdot X) = \xi_t \cdot \tilde{\rho}_t(X)$.

We present a (non-exhaustive) list of axioms for monetary and return risk measures; see, e.g., [9, 18, 30, 38] for further details and discussion. Let $t \in [0,T]$.

Definition 3. *A risk measure $\rho_t : \mathcal{X}(\mathcal{F}_T) \to \mathcal{X}(\mathcal{F}_t)$ is: (i) convex if for any $X, Y \in \mathcal{X}(\mathcal{F}_T)$ and $\lambda \in [0,1]$ such that $\lambda X + (1-\lambda)Y \in \mathcal{X}(\mathcal{F}_T)$ it holds that $\rho_t(\lambda X + (1-\lambda)Y) \leq \lambda\rho_t(X) + (1-\lambda)\rho_t(Y)$; (ii) positively homogeneous if for any $X \in \mathcal{X}(\mathcal{F}_T)$ and $\eta_t \in L_+^\infty(\mathcal{F}_t)$ such that $\eta_t \cdot X \in \mathcal{X}(\mathcal{F}_T)$ it results that $\rho_t(\eta_t \cdot X) = \eta_t \cdot \rho_t(X)$.*

A monotone functional $\tilde{\rho}_t : \mathcal{Y}(\mathcal{F}_T) \to \mathcal{Y}(\mathcal{F}_t)$ is: (i) multiplicatively convex if for any $X, Y \in \mathcal{Y}(\mathcal{F}_T)$ and $\lambda \in [0,1]$ such that $X^\lambda Y^{1-\lambda} \in \mathcal{Y}(\mathcal{F}_T)$ it holds that $\rho_t(X^\lambda Y^{1-\lambda}) \leq \rho_t^\lambda(X)\rho_t^{1-\lambda}(Y)$; (ii) star-shaped if for any $X \in \mathcal{Y}(\mathcal{F}_T)$ and $\eta_t \in \mathcal{L}_+^\infty(\mathcal{F}_t)$ with $\eta_t \leq 1$ $d\mathbb{P}$-a.s. such that $\eta_t \cdot X \in \mathcal{Y}(\mathcal{F}_T)$ it results that $\rho_t(\eta_t \cdot X) \leq \eta_t \cdot \rho_t(X)$.

A risk measure $\rho_t : \mathcal{X}(\mathcal{F}_T) \to \mathcal{X}(\mathcal{F}_t)$ is time-consistent on $\mathcal{X}(\mathcal{F}_T)$ if $\rho_s(X) = \rho_s^t(\rho_t(X))$ for any $t, s \in [0,T]$ with $s < t$ and $X \in \mathcal{X}(\mathcal{F}_T)$ where $\rho_s^t : \mathcal{X}(\mathcal{F}_t) \to \mathcal{X}(\mathcal{F}_s)$ is the restriction of ρ_s to $\mathcal{X}(\mathcal{F}_t)$, i.e., $\rho_s^t := (\rho_s)_{|\mathcal{X}(\mathcal{F}_t)}$. Time-consistency of functionals defined on $\mathcal{Y}(\mathcal{F}_T)$, such as $\tilde{\rho}_t$, can be formulated analogously.

We presented the preceding definition without imposing a linear structure on $\mathcal{X}$ and $\mathcal{Y}$, recognizing that certain spaces we will utilize hereafter may lack this property. A one-to-one correspondence between return risk measures and monetary risk measures has been proved in [9] in a static setting, using as reference spaces $\mathcal{X} \equiv L^\infty$ and $\mathcal{Y} \equiv \mathcal{L}^\infty$. We can extend this one-to-one correspondence to the dynamic case. Starting from a monetary risk measure $\rho_t : L^\infty(\mathcal{F}_T) \to L^\infty(\mathcal{F}_t)$, there is a natural way to define the corresponding return risk measure $\tilde{\rho}_t : \mathcal{L}^\infty(\mathcal{F}_T) \to \mathcal{L}^\infty(\mathcal{F}_t)$ by setting

$$\tilde{\rho}_t(X) := \exp(\rho_t(\ln(X)). \tag{2.1}$$

We notice that, for each $t \in [0,T]$, $\tilde{\rho}_t$ is well-defined by Lemma 1. Indeed, if $X \in \mathcal{L}^\infty(\mathcal{F}_T)$, then $\ln(X) \in L^\infty(\mathcal{F}_T)$, thus $\rho_t(\ln(X)) \in L^\infty(\mathcal{F}_t)$, hence $\exp(\rho_t(\ln(X)) \in \mathcal{L}^\infty(\mathcal{F}_t)$. Conversely, given a return risk measure $\tilde{\rho}_t : \mathcal{L}^\infty(\mathcal{F}_T) \to \mathcal{L}^\infty(\mathcal{F}_t)$, we can define the corresponding monetary risk measure $\rho_t : L^\infty(\mathcal{F}_T) \to L^\infty(\mathcal{F}_t)$ via the formula

$$\rho_t(X) := \ln\left(\tilde{\rho}_t\left(e^X\right)\right). \tag{2.2}$$

Once again, ρ_t is well defined as $e^X \in \mathcal{L}^\infty(\mathcal{F}_T)$ for any $X \in L^\infty(\mathcal{F}_T)$, thus $\tilde{\rho}_t(e^X) \in \mathcal{L}^\infty(\mathcal{F}_t)$ and Lemma 1 ensures that $\ln(\tilde{\rho}_t(X)) \in L^\infty(\mathcal{F}_t)$. Properties of $\tilde{\rho}_t$ and ρ_t are summarized in Proposition A.1.1 in Appendix A.

3 GBSDEs

Let us consider the ensuing BSDE:

$$Y_t = X + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s. \tag{3.1}$$

The pair (X, f) is henceforth referred to as the 'parameters' of the associated BSDE.

Definition 4. *The couple (Y, Z) is a solution to Equation (3.1) if it verifies this equation in the Itô's sense, Y is predictable with continuous trajectories, $Z \in \mathcal{L}_T^2$ is predictable, and $\int_0^T |f(s, Y_s, Z_s)|ds < +\infty$, $d\mathbb{P}$-a.s.*

Throughout this section, we consider the following assumptions on the driver f:

A1) Let $f : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be a $\mathcal{P} \times \mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R}^n)$-measurable function, where $\mathcal{P}$ denotes the σ-algebra generated by predictable sets on $\Omega \times [0,T]$;

A2) There exists $C > 0$ such that, $d\mathbb{P} \times dt$-a.s., for any $(y, z) \in \mathbb{R} \times \mathbb{R}^n$:

$$|f(t, y, z)| \leq C(1 + |y| + |z|^2);$$

A3) There exists $C' > 0$ such that, $d\mathbb{P} \times dt$-a.s., for any $(y_1, y_2, z_1, z_2) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n$:

$$|f(t, y_1, z_1) - f(t, y_2, z_2)| \leq C'[|y_1 - y_2| + (1 + |y_1| + |y_2| + |z_1| + |z_2|)|z_1 - z_2|].$$

Furthermore, we assume that $X \in L^\infty(\mathcal{F}_T)$. Under assumptions A1) and A2), there exist a maximal and a minimal solution to Equation (3.1), and any solution (Y, Z) between the minimal and the maximal solution verifies the regularity $(Y, Z) \in \mathcal{H}_T^\infty \times \mathrm{BMO}(\mathbb{P})$ (see, e.g., [36, 55]). In addition, if A3) is also satisfied, then the solution is unique in the class of solutions belonging to $\mathcal{H}_T^\infty \times \mathrm{BMO}(\mathbb{P})$. Furthermore, for each $t \in [0, T]$, the map $X \mapsto Y_t^X$ is monotone.

3.1 From BSDEs to geometric BSDEs

We are interested in studying the class of return risk measures whose dynamics are induced via certain stochastic differential equations. Let us consider the following BSDE, where for brevity we write ρ_t instead of $\rho_t(\ln(X))$:[2]

$$\rho_t = \ln X + \int_t^T f(s, \rho_s, Z_s)ds - \int_t^T Z_s dW_s. \tag{3.2}$$

Let f satisfy conditions A1) and A2), and $X \in \mathcal{L}^\infty(\mathcal{F}_T)$. Leveraging the existence results referenced above, for each $t \in [0, T]$, the first component of the solution to Equation (3.2) can be interpreted as a function between $L^\infty(\mathcal{F}_T)$ and $L^\infty(\mathcal{F}_t)$, i.e., $\rho_t : L^\infty(\mathcal{F}_T) \to L^\infty(\mathcal{F}_t)$, mapping $\ln(X) \mapsto \rho_t(\ln(X))$. For each fixed $X \in \mathcal{L}^\infty(\mathcal{F}_T)$, we can apply Itô's formula to find the dynamics of the map $t \mapsto \tilde{\rho}_t(X) := \exp(\rho_t(\ln X))$; cf. (2.1). After simple algebra, we find that the dynamics of $\tilde{\rho}_\cdot(X)$ can be described via a *geometric BSDE* (GBSDE) given by

$$\begin{cases} -d\tilde{\rho}_t = \tilde{\rho}_t \left(f(t, \ln(\tilde{\rho}_t), Z_t) - \frac{1}{2}|Z_t|^2 \right) dt - \tilde{\rho}_t Z_t dW_t, \\ \tilde{\rho}_T(X) = X. \end{cases} \tag{3.3}$$

Clearly, Equation (3.3) can also be written as

$$\begin{cases} -d\tilde{\rho}_t/\tilde{\rho}_t = \left(f(t, \ln(\tilde{\rho}_t), Z_t) - \frac{1}{2}|Z_t|^2 \right) dt - Z_t dW_t, \\ \tilde{\rho}_T(X) = X. \end{cases}$$

Here, the multiplicative structure is even more apparent. The corresponding integral form is given by

$$\tilde{\rho}_t = X + \int_t^T \tilde{\rho}_s \left(f(s, \ln(\tilde{\rho}_s), Z_s) - \frac{1}{2}|Z_s|^2 \right) ds - \int_t^T \tilde{\rho}_s Z_s dW_s.$$

3.2 Well-posedness of GBSDEs

The reasoning in the previous subsection suggests the definition, in full generality, of the following GBSDE:

$$\begin{cases} -d\tilde{\rho}_t/\tilde{\rho}_t = \tilde{f}(t, \tilde{\rho}_t, \tilde{Z}_t)dt - \tilde{Z}_t dW_t, \\ \tilde{\rho}_T(X) = X. \end{cases} \tag{3.4}$$

We consider the following assumptions on $\tilde{f}$:

R1) Let $\tilde{f} : \Omega \times [0, T] \times \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ be a $\mathcal{P} \times \mathcal{B}(\mathbb{R}_+) \times \mathcal{B}(\mathbb{R}^n)$-measurable function;

R2) There exists $C > 0$ such that, $d\mathbb{P} \times dt$-a.s., for any $(y, z) \in \mathbb{R} \times \mathbb{R}^n$:

$$|\tilde{f}(t, y, z)| \le C(1 + |\ln y| + |z|^2);$$

R3) There exists $C' > 0$ such that, $d\mathbb{P} \times dt$-a.s., for any $(y_1, y_2, z_1, z_2) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n$:

$$|\tilde{f}(t, y_1, z_1) - \tilde{f}(t, y_2, z_2)| \le C'[|\ln y_1 - \ln y_2| + (1 + |\ln y_1| + |\ln y_2| + |z_1| + |z_2|)|z_1 - z_2|].$$

[2]We will use this shorthand notation also in the sequel without further specification.

Proposition 5. *Under assumptions R1) and R2), if $X \in \mathcal{L}^\infty(\mathcal{F}_T)$, then Equation (3.4) admits a positive solution $(\tilde{\rho}, \tilde{Z}) \in \mathcal{H}_T^\infty \times BMO(\mathbb{P})$. In addition, if R3) holds, the positive solution is unique. Furthermore, for each $t \in [0,T]$, $\tilde{\rho}_t$ is monotone w.r.t. the terminal condition X.*

Example 6. *Consider the driver $\tilde{f} \equiv 0$. Then, we obtain the following GBSDE:*

$$\tilde{\rho}_t(X) = X + \int_t^T \tilde{\rho}_s \tilde{Z}_s dW_s, \;\; X \in \mathcal{L}^\infty(\mathcal{F}_T).$$

This GBSDE admits a unique solution $(\tilde{\rho}, \tilde{Z}) \in \mathcal{H}_T^\infty \times BMO(\mathbb{P})$, by Proposition 5. The explicit form of the first component of the solution is given by $\tilde{\rho}_t(X) = \mathbb{E}[X|\mathcal{F}_t]$, $d\mathbb{P} \times dt$-a.s. This result aligns with the geometric martingale representation theorem (see Proposition 6.4 in [7] and Lemma A.1 in [21]). Thus, the geometric martingale representation theorem occurs as a particular case of a GBSDE when $\tilde{f} \equiv 0$, just like the (additive) martingale representation theorem occurs as a particular case of an ordinary BSDE when $f \equiv 0$.

Example 7. *Consider $\tilde{f}(t,z) = -\frac{1}{2}|z|^2$ and $X \in \mathcal{L}^\infty(\mathcal{F}_T)$. Proceeding as in the proof of Proposition 5, the BSDE with parameters $(\ln(X), 0)$ admits a unique solution $(\rho, Z) \in \mathcal{H}_T^\infty \times BMO(\mathbb{P})$, with $\rho_t(\ln(X)) = \mathbb{E}\left[\ln(X)|\mathcal{F}_t\right]$ $d\mathbb{P} \times dt$-a.s. Hence, the closed-form expression of $(\tilde{\rho}_t)_{t\in[0,T]}$ is given by the following formula:*

$$\tilde{\rho}_t(X) = \exp(\mathbb{E}[\ln(X)|\mathcal{F}_t]), \;\; d\mathbb{P} \times dt\text{-a.s.}$$

Thus, for any $t \in [0,T]$, $X \mapsto \tilde{\rho}_t(X)$ is the geometric conditional expectation, whose dynamics are represented in terms of a GBSDE.

3.3 Using the one-to-one correspondence and going beyond

Next, note that Equation (3.4) can be transformed into an ordinary BSDE by setting $\bar{Z}_t := \tilde{\rho}_t \tilde{Z}_t$. Deploying this substitution, we obtain the following BSDE:

$$\bar{\rho}_t = X + \int_t^T \bar{\rho}_s \tilde{f}(s, \bar{\rho}_s, \bar{Z}_s/\bar{\rho}_s)ds - \int_t^T \bar{Z}_s dW_s.$$

Thus, we can define a new driver $\bar{f}(t,y,z) := \tilde{f}(t,y,z/y)$ that satisfies a *logarithmic-quadratic* (LN-Q) growth rate of the form

$$y\bar{f}(t,y,z) \leq C(y + y|\ln(y)| + |z|^2/y), \;\; \forall (y,z) \in \mathbb{R}_+ \times \mathbb{R}^n.$$

Since we want to allow also for stochastic and unbounded coefficients, we consider the following generalized growth rate:

$$y\bar{f}(t,y,z) \leq \alpha_t y + \beta_t y|\ln(y)| + \gamma_t |z| + \delta |z|^2/y, \;\; \forall (y,z) \in \mathbb{R}_+ \times \mathbb{R}^n,$$

where the term $\gamma_.|z|$ has been added for the sake of generality. Here, α, β, γ are predictable and positive stochastic processes and $\delta \geq 0$ is constant. Motivated by this observation, we consider in the following an 'ordinary' BSDE

$$Y_t = X + \int_t^T g(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s, \tag{3.5}$$

where $g : [0,T] \times \Omega \times \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ is a $\mathcal{P} \times \mathcal{B}(\mathbb{R}_+) \times \mathcal{B}(\mathbb{R}^n)/\mathcal{B}(\mathbb{R})$-measurable function with LN-Q growth rate

$$|g(t,y,z)| \leq \alpha_t y + \beta_t y|\ln(y)| + \gamma_t|z| + \delta|z|^2/y, \;\; d\mathbb{P} \times dt\text{-a.s. } \forall (y,z) \in \mathbb{R}_+ \times \mathbb{R}^n. \tag{3.6}$$

The next proposition shows that it is always possible to find a solution to a BSDE whose driver has an LN-Q growth rate once we are provided with a solution to a quadratic BSDE

$$Y'_t = X' + \int_t^T g'(s, Y'_s, Z'_s)ds - \int_t^T Z'_t dW_s, \tag{3.7}$$

with

$$|g'(t, y, z)| \le \alpha_t + \beta_t |y| + \gamma_t |z| + \eta |z|^2, d\mathbb{P} \times dt\text{-a.s. } \forall (y, z) \in \mathbb{R} \times \mathbb{R}^n, \tag{3.8}$$

where $\eta \ge 0$, and *vice versa*. This proposition is in the same spirit of Proposition 5, taking into account a more general driver.

Proposition 8. *Let g satisfy condition (3.6), where, α, β, γ are predictable and positive stochastic processes.*

If Equation (3.5) admits a positive solution with parameters (X, g) where $X \in L^0_+(\mathcal{F}_T)$, then Equation (3.7) admits a solution with parameters (X', g'), where g' verifies Equation (3.8) with $\eta = \delta + 1/2$ and $X' \in L^0(\mathcal{F}_T)$.

Conversely, assume that Equation (3.7) admits a solution with parameters (X', g'), where g' verifies Equation (3.8) and $X' \in L^0(\mathcal{F}_T)$. Then, Equation (3.5) admits a positive solution with parameters (X, g), where g satisfies Equation (3.6) with $\delta = \eta + 1/2$ and $X \in L^0_+$.

Corollary 9. *With the same notation as in Proposition 8, let $X \in \mathcal{L}^\infty(\mathcal{F}_T)$ and $\alpha, \beta, \gamma \in \mathcal{H}^\infty_T$. Then, there exist a maximal and minimal solution to Equation (3.5) with parameters (X, g). Each solution between the minimal and the maximal verifies $(Y, Z) \in \mathcal{H}^\infty_T \times BMO(\mathbb{P})$.*

As is evident from the proof of Corollary 9, the restriction $X \in \mathcal{L}^\infty(\mathcal{F}_T)$ is *pivotal* so far. Without this assumption, establishing the existence and properties of a solution becomes more challenging and mathematically involved, since then the terminal condition $X' = \ln(X)$ can be unbounded and may even not be integrable. To investigate the existence, regularity, uniqueness and stability of the solution in broader spaces, we must explore genuinely distinct approaches, *not* invoking one-to-one correspondences as in Corollary 9. In addition, because we want to allow for coefficients α, β, γ that are not necessarily bounded (or constant), we must employ proof strategies different from those utilized above to establish existence and regularity of the solution. These and related problems are studied in the following section.

4 Main Results for LN-Q BSDEs

In this section, we study ordinary BSDEs of the form given in Equation (3.5) with drivers satisfying the LN-Q growth rate in (3.6). As is clear from Section 3.3, it is non-trivial to go beyond the one-to-one correspondence and relax the assumptions of bounded terminal conditions (and bounded coefficients), which will be the aim of this section. For the sake of generality and completeness, we consider the following assumptions:

(H1) $g : \Omega \times [0, T] \times \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ is a predictable stochastic process, continuous in (y, z) $d\mathbb{P} \times dt$-a.s., verifying

$$0 \le g(t, y, z) \le \alpha_t y + \beta_t y |\ln(y)| + \delta |z|^2 / y =: h(t, y, z),$$

where α, β are positive and predictable stochastic processes and $\delta > 0$;

(H1)' With the same notation as in (H1), the process g now verifies

$$0 \le g(t, y, z) \le \alpha_t y + \beta_t y |\ln(y)| + \gamma_t |z| + \delta |z|^2 / y =: h(t, y, z),$$

where $\gamma : [0, T] \times \Omega \to \mathbb{R}$ is a positive and predictable stochastic process;

(H1)" With the same notation as in (H1), the process g now verifies

$$0 \le g(t,y,z) \le \alpha_t y + \beta_t y|\ln(y)| + \eta_t \cdot z + \delta|z|^2/y =: h(t,y,z),$$

where $\eta : [0,T] \times \Omega \to \mathbb{R}^n$ is a predictable stochastic process;

(H2) Let X be a strictly positive $\mathcal{F}_T$-measurable random variable such that

$$\mathbb{E}\left[(1+X^{2\delta+1})^{e^B} \exp\left(e^B(A+B)\right)\right] < +\infty,$$

where $A := \int_0^T \alpha_t dt$ and $B := \int_0^T \beta_t dt$.

(H2)' With the same notation as in (H2), let $p > 1$ such that

$$\mathbb{E}\left[(1+X^{2\delta+1})^{p(e^B+1)} \exp\left(p(e^B+1)(A+B)\right)\right] < +\infty.$$

In addition, we assume $\alpha, \beta \in \mathcal{H}_T^q$, with $\frac{1}{q} + \frac{1}{p} = 1$.

(H2)" With the same notation as in (H2), let $p > 1$ and γ be as in (H1)' such that

$$\mathbb{E}\left[(1+X^{2\delta+1})^{p(e^B+1)} \exp\left(p(e^B+1)\left((A+B) + \frac{1}{4\eta}\int_0^T \gamma_t^2 dt\right)\right)\right] < +\infty.$$

Here, $\eta \in (0, \frac{1\wedge(p-1)}{2})$. Furthermore, we assume there exists $q' > 0$ such that $\mathbb{E}[\int_0^T e^{q'\gamma_t} dt] < +\infty$ and $\alpha, \beta \in \mathcal{H}_T^q$, with $\frac{1}{q} + \frac{1}{p} = 1$.

We note that we employ one and the same $h(\cdot,\cdot,\cdot)$ for different functions across (H1), (H1)', and (H1)", with slight abuse of notation. In the results that follow, the respective form of h will be specified on a case-by-case basis. Although certain assumptions from the previous set clearly imply others, we opt to address them individually to highlight the distinct regularities associated with each case.

4.1 Existence and regularity with unbounded terminal conditions and coefficients

We start by establishing existence and regularity results in the setting of L^p-spaces.

Proposition 10. *(i) Assume (H1) and (H2). Then, Equation (3.5) with driver h as defined in (H1) admits a positive solution such that*

$$\sup_{t\in[0,T]} \mathbb{E}\left[Y_t^{(2\delta+1)e^B}\right] < +\infty, \quad \text{and } Z \in \mathcal{L}_T^2.$$

(ii) Assume (H1) and (H2)'. Then, there exists a positive solution to Equation (3.5) with driver h as defined in (H1), verifying the further regularity

$$\mathbb{E}\left[\sup_{t\in[0,T]} Y_t^{p(2\delta+1)(e^B+1)}\right] < +\infty, \quad \text{and } Z \in \mathcal{M}_T^2.$$

(iii) Assume (H1)' and (H2)". Then, there exists a positive solution to Equation (3.5) with driver h as defined in (H1)', verifying the same regularity as in the case of (H1)+(H2)'.

Theorem 11. *Assume (H1) and (H2), with driver g as in (H1). Then, there exists a positive solution to Equation (3.5) in the sense of Definition 4 verifying $0 < Y \le Y^h$, where Y^h is a solution to Equation (3.5) with driver h. We have the regularity $0 < \sup_{t\in[0,T]} \mathbb{E}\left[Y_t^{(2\delta+1)e^B}\right] < +\infty$ and $Z \in \mathcal{L}_T^2$. Furthermore, if we assume (H1) and (H2)' (or, alternatively, (H1)' and (H2)"), then the solution has the further regularity $0 < \mathbb{E}\left[\sup_{t\in[0,T]} Y_t^{p(2\delta+1)(e^B+1)}\right] < +\infty$ and $Z \in \mathcal{M}_T^2$.*

Proof. We first wish to apply Lemma A.2.1 in Appendix A. We choose $X_1 = X_2 = X$, $g_1 \equiv 0$ and $g_2(t, y, z) := h(t, y, z) = \alpha_t y + \beta_t y|\ln(y)| + \delta|z|^2/y$. Clearly, $g_1 \equiv 0 \leq g \leq g_2$. Furthermore, the BSDE with parameters $(X, 0)$ admits a unique positive solution Y^0 according to Proposition 1.1 (i) in [2], while Proposition 10 ensures the existence of a positive solution Y^h to the BSDE with parameters (X, h). In addition, it holds that $0 < Y^0 \leq Y^h$. Indeed, after a suitable localization $(\tau_n)_{n\in\mathbb{N}}$, we obtain

$$\mathbb{E}[Y_t^h|\mathcal{F}_t] = \mathbb{E}\left[X + \int_t^{\tau_n} g_2(s, Y_s^h, Z_s^h)ds \middle| \mathcal{F}_t\right] \geq \mathbb{E}[X|\mathcal{F}_t] = Y_t^0,$$

where the inequality follows by positivity of h. Finally, for any $(t, \omega) \in [0, T] \times \Omega$ and $y \in [Y_t^0(\omega), Y_t^h(\omega)]$, it results that

$$g(t,y,z) \leq (\alpha_t + \beta_t) + (\alpha_t + \beta_t)(1+|y|^2) + \frac{\delta|z|^2}{y} \leq (\alpha_t + \beta_t) + (\alpha_t+\beta_t)(1+|Y_t^{g_2}(\omega)|^2) + \frac{\delta|z|^2}{Y_t^0(\omega)},$$

thus g satisfies all the conditions of Lemma A.2.1. This ensures the existence of a solution (Y, Z) to the BSDE with parameters (X, g), subject to the additional constraint $0 < Y^0 \leq Y \leq Y^h$. Clearly, the last inequality implies the desired regularities for Y. The cases (H1)+(H2)' and (H1)'+(H2)" can be proved similarly.

We will now establish the inclusion of Z in $\mathcal{M}_T^2$ under the combined hypotheses (H1)'+(H2)", as (H1)+(H2)' is a specific instance of this case. We need a proof strategy different from that employed in Proposition 10. Let us consider the function $f(x) = x^\eta$, where $\eta > 0$ is a parameter that will be determined later. We have that

$$\begin{aligned}
Y_t^\eta &= X^\eta + \int_t^T \left(\eta Y_s^{\eta-1} g(s, Y_s, Z_s) - \frac{\eta(\eta-1)}{2} Y_s^{\eta-2}|Z_s|^2\right) ds - \int_t^T \eta Y_s^{\eta-1} Z_s dW_s \\
&\leq X^\eta + \int_t^T \left[\eta Y_s^{\eta-1}\left(\alpha_s Y_s + \beta_s Y_s|\ln(Y_s)| + \gamma_s|Z_s| + \delta\frac{|Z_s|^2}{Y_s}\right) - \frac{\eta(\eta-1)}{2} Y_s^{\eta-2}|Z_s|^2\right] ds \\
&- \int_t^T \eta Y_s^{\eta-1} Z_s dW_s \\
&= X^\eta + \int_t^T \eta(\alpha_s Y_s^\eta + \beta_s Y_s^\eta|\ln(Y_s)| + \gamma_s Y_s^{\eta-1}|Z_s|)ds \\
&+ \int_t^T \eta\left(\delta - \frac{\eta-1}{2}\right) Y_s^{\eta-2}|Z_s|^2 ds - \int_t^T \eta Y_s^{\eta-1} Z_s dW_s.
\end{aligned}$$

Taking $\eta > 2\delta + 1$, $t = 0$ and upon rearranging, we obtain

$$\begin{aligned}
&\int_0^T \eta\left(\frac{\eta-1}{2} - \delta\right) Y_s^{\eta-2}|Z_s|^2 ds \\
&\leq -Y_0^\eta + X^\eta + \int_0^T \eta Y_s^{\eta-1}(\alpha_s Y_s + \beta_s Y_s|\ln(Y_s)| + \gamma_s|Z_s|)ds - \int_0^T \eta Y_s^{\eta-1} Z_s dW_s. \qquad (4.1)
\end{aligned}$$

We start by proving the thesis when Y is sufficiently large. In this case, we must require that $Y^{\eta-2}$ does not go to zero, which is guaranteed by imposing $\eta \geq 2$. Considering a sequence of stopping times as in Proposition 10, taking the expectation in Equation (4.1), applying Young's inequality and using the fact that $Y^{\eta-1}|Z| = Y^{\eta/2} Y^{(\eta-2)/2}|Z|$, we obtain that

$$\begin{aligned}
&\mathbb{E}\Big[\int_0^{\tau_n} \eta\left(\frac{\eta-1}{2} - \delta\right) Y_s^{\eta-2}|Z_s|^2 ds\Big] \\
&\leq \mathbb{E}\left[-Y_0^\eta + X^\eta\right] \\
&+ \mathbb{E}\left[\int_0^T \left(\frac{(\eta\alpha_s)^q}{q} + \frac{Y_s^{p\eta}}{p} + \frac{(\eta\beta_s)^q}{q} + \frac{Y_s^{p\eta}|\ln^p(Y_s)|}{p} + \frac{(\eta\gamma_s Y_s^{\eta/2})^2}{2\varepsilon} + \frac{\varepsilon}{2} Y_s^{\eta-2}|Z_s|^2\right) ds\right].
\end{aligned}$$

Let us consider $\frac{\varepsilon}{2} < \eta(\frac{\eta-1}{2} - \delta)$. Then, it holds that

$$\begin{aligned}
&K\mathbb{E}\left[Y_s^{\eta-2}\int_0^T |Z_s|^2 ds\right] \\
&\le \mathbb{E}\left[-Y_0^\eta + X^\eta + \int_0^T \frac{(\eta\alpha_s)^q}{q} + \frac{Y_s^{p\eta}}{p} + \frac{(\eta\beta_s)^q}{q} + \frac{Y_s^{p\eta}|\ln^p(Y_s)|}{p} + \frac{(\eta\gamma_s)^{2q}}{2\varepsilon q} + \frac{Y_s^{p\eta}}{2\varepsilon p} ds\right] \\
&\le K'\left(1 + \mathbb{E}\left[\sup_{t\in[0,T]} Y_t^{\eta p + \varepsilon'}\right]\right),
\end{aligned}$$

where $K := \eta(\frac{\eta-1}{2} - \delta) - \frac{\varepsilon}{2} > 0$. The first inequality follows from Young's inequality, while the second inequality is implied by $y^m|\ln^m(y)| \le K_{\varepsilon',m} + y^{m+\varepsilon'}$, holding for any $m > 1$, and $\varepsilon' > 0$ for some suitable constant $K_{\varepsilon',m} > 0$. In particular, $K' > 0$ is a constant depending only on $T, \alpha, \beta, \gamma, \eta, p, q, \varepsilon$ and on $\varepsilon' > 0$ (which will be fixed in the following).

Recalling the regularity of Y, we have $\mathbb{E}[\sup_{t\in[0,T]} Y_t^{p(2\delta+1)(e^B+1)}] < +\infty$, where $e^B + 1 \ge 2$ $\mathbb{P}$-a.s. Thus, we impose $\eta p + \varepsilon' = 2p(2\delta + 1)$. Hence, $0 < \varepsilon' = 2p(2\delta + 1) - \eta p$ yielding $\eta < 2(2\delta + 1)$. In sum, we have the following conditions on η:

$$\begin{cases} \eta < 4\delta + 2, \\ \eta \ge 2, \\ \eta > 2\delta + 1. \end{cases}$$

After simple algebra, it is possible to check that for any value of $\delta \in \mathbb{R}_+$ there exists $\eta > 0$ verifying all the above conditions. Thus, we obtain $\mathbb{E}\left[\int_0^T Y_s^{\eta-2}|Z_s|^2 ds\right] < +\infty$. Let $C > 0$ be a positive constant and define $A := \{(\omega, t) \in \Omega \times [0, T] : \ Y_t(\omega) \ge C\}$. Then, we have

$$C^{\eta-2}\mathbb{E}\left[\int_0^T |Z_s|^2 \mathbb{I}_A ds\right] \le \mathbb{E}\left[\int_0^T Y_s^{\eta-2}|Z_s|^2 ds\right] < +\infty. \tag{4.2}$$

Now, we prove the integrability when Y is sufficiently small. Consider the substitution $f(x) := \ln(1 + x)$. By Itô's formula it holds that

$$\ln(1 + Y_t) = \ln(1 + X) + \int_t^T \frac{1}{1 + Y_s} g(s, Y_s, Z_s) + \frac{1}{2(1 + Y_s)^2}|Z_s|^2 ds - \int_t^T \frac{1}{1 + Y_s} Z_s dW_s.$$

Since $g \ge 0$ and $Y > 0$, we have

$$\ln(1 + Y_t) \ge \ln(1 + X) + \int_t^T \frac{1}{2(1 + Y_s)^2}|Z_s|^2 ds - \int_t^T \frac{1}{1 + Y_s} Z_s dW_s.$$

Upon rearranging, we obtain

$$\int_t^T \frac{1}{2(1 + Y_s)^2}|Z_s|^2 ds \le \ln(1 + Y_t) - \ln(1 + X) + \int_t^T \frac{1}{1 + Y_s} Z_s dW_s,$$

and hence the same inequality when $t = 0$. Define $\tau_n := \inf\{t > 0 : \int_0^t \frac{1}{(1+Y_s)^2}|Z_s|^2 ds \ge n\} \wedge T$. Clearly, for each $n \in \mathbb{N}$, τ_n is a stopping time, and $\tau_n \to T$ $d\mathbb{P}$-a.s. By employing this localization, taking the expectation and recalling the inequality $\ln(1 + x) \le x$ for any $x \ge 0$, we obtain

$$\mathbb{E}\left[\int_0^{\tau_n} \frac{1}{2(1 + Y_s)^2}|Z_s|^2 ds\right] \le Y_0 + \mathbb{E}[X] < +\infty.$$

Letting $n \to \infty$, Fatou's lemma leads to

$$\mathbb{E}\left[\int_0^T \frac{1}{2(1 + Y_s)^2}|Z_s|^2 ds\right] \le Y_0 + \mathbb{E}[X] < +\infty.$$

This inequality yields

$$\frac{1}{2(1+C)^2}\mathbb{E}\left[\int_0^T |Z_s|^2 \mathbb{I}_{A^c} ds\right] \leq Y_0 + \mathbb{E}[X] < +\infty. \tag{4.3}$$

Upon combining Equations (4.2) and (4.3), the thesis follows. □

The following corollary provides sufficient conditions under which Equation (3.5) admits a unique solution, considering a driver of a specific form.

Corollary 12. *Assume (H1)' with $h(t,y,z) = \alpha_t y + \beta_t y|\ln(y)| + \gamma_t|z| + \delta|z|^2/y$. If $\alpha, \beta, \gamma \in \mathcal{H}_T^\infty$ and there exists $\lambda \geq 2\|\beta\|_\infty^T$ such that $\mathbb{E}[X^{(2\delta+1)(e^{\lambda T}+1)}] < +\infty$, then Equation (3.5) admits a unique solution with regularity $(Y,Z) \in \mathcal{H}_T^{(2\delta+1)(e^{\lambda T}+1)} \times \mathcal{M}_T^2$.*

Remark 13. *Note that both Proposition 10 and Theorem 11 also establish the existence of maximal and minimal solutions to Equation (3.5) driven by h and g, respectively. The existence of these solutions can be inferred from Lemma A.2.1.*

4.2 Further regularities in the bounded case

Additional regularities can be established under boundedness conditions, as follows.

Proposition 14. *Assume $X \in L_+^\infty(\mathcal{F}_T)$, $\alpha, \beta, \gamma \in \mathcal{H}_T^\infty$ and $\delta > 0$. Let h be as defined in Corollary 12. Then the BSDE*

$$Y_t^h = X + \int_t^T h(s, Y_s^h, Z_s^h) ds - \int_t^T Z_s^h dW_s, \tag{4.4}$$

admits a unique solution $(Y^h, Z^h) \in \mathcal{H}_T^\infty \times BMO(\mathbb{P})$. Furthermore, under the same hypotheses on the coefficients, if $g : [0,T] \times \Omega \times \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ verifies (H1)', then the corresponding BSDE admits at least one solution $(Y,Z) \in \mathcal{H}_T^\infty \times BMO(\mathbb{P})$. Specifically, it admits maximal and minimal solutions with such regularity, among all possible solutions verifying $0 < Y \leq Y^h$.

Corollary 15. *Consider $X \in L_+^\infty(\mathcal{F}_T)$, $\alpha, \beta \in \mathcal{H}_T^\infty$ and $\gamma \in BMO(\mathbb{P})$, and let $g : [0,T] \times \Omega \times \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ verify (H1)". Then, the BSDE with parameters (X,g) admits at least one solution. Furthermore, any solution is such that $(Y,Z) \in \mathcal{H}_T^\infty \times BMO(\mathbb{P})$.*

4.3 Uniqueness

In this subsection, we study uniqueness of the solution to Equation (3.5). We require a further assumption on the driver g.

C) The driver g is jointly convex in $(y,z) \in \mathbb{R}_+ \times \mathbb{R}^n$.

In [6], uniqueness under a $|z|^2/y$-growth condition is studied, for which the authors impose convexity of g and require the terminal condition X to be bounded and bounded away from zero. Their proof strategy relies on exploiting these properties to derive dual representations. Our approach is different: it is based on proving a comparison theorem that implies uniqueness as a consequence. In [5], uniqueness under $y|\ln(y)|$-growth is analyzed. There, the authors introduce a monotonicity condition instead of requiring convexity, and they employ an entirely different strategy for their proofs. Our current investigation establishes the existence of unique solutions within the framework of a driver exhibiting a growth rate involving $y|\ln(y)| + |z|^2/y$. Our setting encompasses the case of unbounded terminal conditions that are not necessarily bounded away from zero. Hence, as a byproduct, the following results expand the theory developed in [6], where only the singularity $|z|^2/y$ is considered under more restrictive assumptions.

Before stating our main results, we require an additional assumption on the driver g:

A) $0 \le g(t,y,z) \le \alpha_t y + \beta_t y|\ln(y)| + \gamma_t \cdot z + \delta|z|^2/y := h(t,y,z), \quad \forall (y,z) \in \mathbb{R}_+ \times \mathbb{R}^n$, where $\alpha, \beta, \gamma \in \mathcal{H}_T^\infty$ and $\delta > 0$.

We note that the bound on g in assumption A) can also be written as

$$0 \le \mathbb{I}_{\{y>0\}} \cdot g(t,|y|,z) \le \mathbb{I}_{\{y>0\}} \cdot (\alpha_t y^+ + \beta_t y^+|\ln(y^+)| + \gamma_t \cdot z + \delta|z|^2/|y|), \quad \forall (y,z) \in \mathbb{R} \times \mathbb{R}^n,$$

with the convention that all members are 0 when $y = 0$.

We start by proving a useful lemma. It can be regarded as a generalization of a stochastic Grönwall lemma (see [33, 54]), accounting for the non-linearity $y|\ln(y)|$.

Lemma 16. *Consider a positive process $\beta \in \mathcal{H}_T^\infty$ and $X \in L_+^{p(e^B+1)}(\mathcal{F}_T)$ for some $p > 1$, where $B := \|\beta\|_\infty^T \cdot T$. Let $u \in \mathcal{H}_T^{p(e^B+1)}$ such that $u_t \ge 0$ $d\mathbb{P} \times dt$-a.s. If u verifies $d\mathbb{P}$-a.s. for any $t \in [0,T]$:*

$$u_t \le \mathbb{E}\left[X + \int_t^T \beta_s u_s |\ln(u_s)| ds \middle| \mathcal{F}_t \right],$$

then there exists an increasing function $\psi : \mathbb{R}_+ \to \mathbb{R}$ such that $d\mathbb{P}$-a.s. for any $t \in [0,T]$:

$$u_t \le \mathbb{E}\left[\psi^{-1}\left(\psi(X) + \int_t^T \beta_s ds \right) \middle| \mathcal{F}_t \right],$$

where $\psi^{-1} : Range(\psi) \to \mathbb{R}_+$ is the inverse function of ψ. The function ψ is given by the explicit formula

$$\psi(x) := \begin{cases} \frac{x-2}{\ln(4)} & \text{if } 0 \le x \le 2, \\ \ln(\ln(x)) - \ln(\ln(2)) & \text{if } x > 2. \end{cases}$$

Theorem 17. *Assuming the driver g satisfies assumptions A), C), and $X \in L_+^{p(2\delta+1)(e^B+1)}$, where $p > \max\{\frac{2}{2\delta+1}, 1\}$ and $B = (\|\alpha\|_\infty^T + \|\beta\|_\infty^T)T$, we consider the BSDE (3.5) with parameters (X,g). Under these conditions, there exists a unique solution $(Y,Z) \in \mathcal{H}_T^{p(2\delta+1)(e^B+1)} \times \mathcal{M}_T^2$ such that $0 < Y \le Y^h$, where Y^h is the (maximal) solution corresponding to the driver h, with the same notation as in assumption A).*[3]

The main tool to prove Theorem 17 is given by the following comparison principle.

Proposition 18. *With the same notation as in Theorem 17, consider a driver g (resp. g') verifying assumptions A), C) and $X, X' \in L_+^{p(2\delta+1)(e^B+1)}$ with $X \le X'$. If $(Y,Z), (Y',Z') \in \mathcal{H}_T^{p(2\delta+1)(e^B+1)} \times \mathcal{M}_T^2$ are solutions to the BSDE with parameters (X,g) and (X',g'), respectively, such that $0 < Y, Y' \le Y^h$, and if the drivers verify*

$$g(t, Y_t', Z_t') \le g'(t, Y_t', Z_t') \quad (\text{resp. } g(t, Y_t, Z_t) \le g'(t, Y_t, Z_t)) \ d\mathbb{P} \times dt\text{-a.s.},$$

then, $d\mathbb{P}$-a.s., $Y_t \le Y_t'$ for any $t \in [0,T]$.

Proof. We assume that g verifies the assumptions A), C) and $g(t, Y_t', Z_t') \le g'(t, Y_t', Z_t')$. The case with g' verifying A), C) and $g(t, Y_t, Z_t) \le g'(t, Y_t, Z_t)$ can be proved similarly. Fix $\theta \in (0,1)$. The proof strategy is to estimate the difference $Y - \theta Y'$ and then let $\theta \to 1$. Define $P := \frac{Y - \theta Y'}{1-\theta}$ and $V := \frac{Z - \theta Z'}{1-\theta}$. It holds that

$$P_t = P_T + \int_t^T G(s, P_s, V_s) ds - \int_t^T V_s dW_s,$$

[3] Here, "unique solution" means that if (Y,Z) and (Y',Z') are two solutions to Equation (3.5) with $0 < Y, Y' \le Y^h$, then Y and Y' are indistinguishable processes, and $Z_t = Z_t'$ $d\mathbb{P} \times dt$-a.s.

where

$$G(\omega,t,y,z):=\frac{1}{1-\theta}\left[g(\omega,t,(1-\theta)y+\theta Y_t'(\omega),(1-\theta)z+\theta Z_t'(\omega))-\theta g(\omega,t,Y_t'(\omega),Z_t'(\omega))\right]$$
$$+\frac{\theta}{1-\theta}\left[g(\omega,t,Y_t'(\omega),Z_t'(\omega))-g'(\omega,t,Y_t'(\omega),Z_t'(\omega))\right],$$

if $((1-\theta)y+\theta Y_t'(\omega),(1-\theta)z+\theta Z_t'(\omega))\in\mathbb{R}_+\times\mathbb{R}^n$ and $G(\omega,t,y,z):=0$ otherwise. By the assumptions on g and g', it can be checked by convexity that $\mathbb{I}_{\{y>0\}}G(t,y,z)\le\mathbb{I}_{\{y>0\}}g(t,|y|,z)$, with the convention that the right-hand member of the previous inequality is zero when $y=0$. Now, we find the dynamics of P^+, which is the positive part of P. Employing Itô-Tanaka's formula, we obtain

$$P_t^+=P_T^++\int_t^T\mathbb{I}_{\{P_s>0\}}G(s,P_s,V_s)ds-\int_t^T\mathbb{I}_{\{P_s>0\}}V_sdW_s-\frac{1}{2}\int_t^TdL_s,$$

where $(L_t)_{t\in[0,T]}$ is the local time at 0 of $(W_t)_{t\in[0,T]}$. We consider $\eta\ge2$, and the function $f(x):=x^\eta$. Itô-Tanaka's formula applied to $f(P_t^+)$ yields

$$f(P_t^+)=f(P_T^+)+\int_t^T\eta(P_s^+)^{\eta-1}\left[\mathbb{I}_{\{P_s>0\}}G(s,P_s,V_s)-\frac{1}{2}(\eta-1)\frac{|V_s|^2}{P_s^+}\right]ds$$
$$-\frac{\eta}{2}\int_t^T(P_s^+)^{\eta-1}dL_s-\int_t^T\eta\mathbb{I}_{\{P_s>0\}}(P_s^+)^{\eta-1}V_sdW_s.$$

Observing that the third term in the right-hand member is non-positive and that by assumption A) it holds that

$$\mathbb{I}_{\{P_t>0\}}G(t,P_t,V_t)\le\mathbb{I}_{\{P_t>0\}}g(t,|P_s|,V_t)\le\mathbb{I}_{\{P_t>0\}}\left(\alpha_tP_t^++\beta_tP_t^+|\ln(P_t^+)|+\gamma_t\cdot V_t+\delta\frac{|V_t|^2}{|P_t|}\right),$$

we obtain

$$f(P_t^+)\le f(P_T^+)$$
$$+\int_t^T\eta(P_s^+)^{\eta-1}\left[\mathbb{I}_{\{P_s>0\}}\left(\alpha_sP_s^++\beta_sP_s^+|\ln(P_s^+)|+\gamma_s\cdot V_s+\delta\frac{|V_s|^2}{|P_s|}\right)-\frac{1}{2}(\eta-1)\frac{|V_s|^2}{P_s^+}\right]ds$$
$$-\int_t^T\eta\mathbb{I}_{\{P_s>0\}}(P_s^+)^{\eta-1}V_sdW_s.$$

Choosing $\eta=\max\{2\delta+1,2\}$, we find that

$$\delta\mathbb{I}_{\{P_t>0\}}(P_t^+)^{\eta-1}|P_t|^{-1}|V_t|^2-\frac{1}{2}(\eta-1)(P_t^+)^{\eta-2}|V_t|^2\le0.$$

Hence,

$$f(P_t^+)\le f(P_T^+)+\int_t^T\eta(P_s^+)^\eta(\alpha_s+\beta_s|\ln(P_s^+)|)+\eta\mathbb{I}_{\{P_s>0\}}(P_s^+)^{\eta-1}\gamma_s\cdot V_sds$$
$$-\int_t^T\eta\mathbb{I}_{\{P_s>0\}}(P_s)^{\eta-1}V_sdW_s$$
$$=f(P_T^+)+\int_t^T\eta(P_s^+)^\eta(\alpha_s+\beta_s|\ln(P_s^+)|)ds-\int_t^T\eta\mathbb{I}_{\{P_s>0\}}(P_s)^{\eta-1}V_sdW_s^\gamma,\qquad(4.5)$$

where we used that $\mathbb{I}_\cdot\le1$ and Girsanov's Theorem since $\gamma\in H_T^\infty$. Fixing $t\in[0,T]$, let us introduce the localization

$$\tau_n:=\inf\left\{s\ge t:\int_t^s(\eta\mathbb{I}_{\{P_u>0\}}(P_u^+)^{\eta-1}V_u)^2du\ge n\right\}\wedge T.$$

Taking the $\mathbb{Q}^\gamma$-conditional expectation in Equation (4.5) and using $x^\eta \le e^\eta + x^\eta|\ln(x)|$, we obtain

$$\begin{aligned}(P_t^+)^\eta &\le \mathbb{E}_{\mathbb{Q}^\gamma}\left[(P_{\tau_n}^+)^\eta + \int_0^{\tau_n} e^\eta \alpha_s ds + \int_t^{\tau_n} (\alpha_s+\beta_s)(P_s^+)^\eta |\ln(P_s^+)^\eta|)ds \middle| \mathcal{F}_t\right] \\ &\le \mathbb{E}_{\mathbb{Q}^\gamma}\left[\left(\sup_{t\in[0,T]} P_t^+\right)^\eta + \int_0^{T} e^\eta \alpha_s ds + \int_t^{\tau_n} (\alpha_s+\beta_s)(P_s^+)^\eta |\ln(P_s^+)^\eta|)ds \middle| \mathcal{F}_t\right].\end{aligned}$$

Here, the conditional expectation w.r.t. $\mathbb{Q}^\gamma$ is well-defined. Indeed, as is well-known, if $\gamma \in \mathcal{H}_T^\infty$, the density $\mathcal{E}_T^\gamma \in L^m(\mathcal{F}_T)$ for any $m \ge 1$. By regularity of Y, Y', we infer that $P \in \mathcal{H}^{p(2\delta+1)(e^B+1)}$ with $p > \max\{\frac{2}{2\delta+1}, 1\}$, thus $(P^+)^\eta \in \mathcal{H}_T^{p'(e^B+1)}$ for some $p' > 1$. Similarly as done in Lemma 16, it is possible to verify that

$$\Gamma := \psi^{-1}\left(\psi\left(\left(\sup_{t\in[0,T]} P_t^+\right)^\eta + \int_0^T e^\eta \alpha_s ds\right) + \int_0^T (\alpha_s+\beta_s)ds\right) \in L^q,$$

for some $q > 1$. Choosing $q' > 0$ such that $\frac{1}{q} + \frac{1}{q'} = 1$, by Young's inequality it holds that

$$\begin{aligned}&\mathbb{E}\left[\mathcal{E}_T^\gamma \psi^{-1}\left(\psi\left((P_{\tau_n}^+)^\eta + \int_0^T e^\eta \alpha_s ds\right) + \int_t^{\tau_n} (\alpha_s+\beta_s)ds\right)\right] \\ \le& C\left(\mathbb{E}\left[(\mathcal{E}_T^\gamma)^{q'}\right] + \mathbb{E}\left[\psi^{-1}\left(\psi\left(\left(\sup_{t\in[0,T]} P_t^+\right)^\eta + \int_0^T e^\eta \alpha_s ds\right) + \int_0^T (\alpha_s+\beta_s)ds\right)^q\right]\right) \\ <& +\infty,\end{aligned}$$

where $C > 0$ is a constant depending only on q, q' and we have used the increasing monotonicity of ψ and ψ^{-1}. Hence, the conditional expectation w.r.t. $\mathbb{Q}^\gamma$ is well-defined. Employing again that $\mathcal{E}_T^\gamma \in L^m(\mathcal{F}_T)$ for any $m \ge 1$, we can check that $P \in \mathcal{H}_T^{l(e^B+1)}(\mathbb{Q}^\gamma)$ for some $1 < l < p'$. Thus, all assumptions of Lemma 16 are verified, yielding

$$(P_t^+)^\eta \le \mathbb{E}_{\mathbb{Q}^\gamma}\left[\psi^{-1}\left(\psi\left((P_{\tau_n}^+)^\eta + \int_0^{\tau_n} e^\eta \alpha_s ds\right) + \int_t^{\tau_n} (\alpha_s+\beta_s)ds\right) \middle| \mathcal{F}_t\right].$$

Since $\tau_n \to T$ a.s. and employing the dominated convergence theorem for conditional expectations, we can let $n \to \infty$ in the previous inequality to obtain

$$(P_t^+)^\eta \le \mathbb{E}_{\mathbb{Q}^\gamma}\left[\psi^{-1}\left(\psi\left((P_T^+)^\eta + \int_0^T e^\eta \alpha_s ds\right) + \int_t^T (\alpha_s+\beta_s)ds\right) \middle| \mathcal{F}_t\right].$$

Observing that $(P_T^+)^\eta = \left(\left(\frac{X-\theta X'}{1-\theta}\right)^+\right)^\eta \le |X|^\eta$ and recalling that ψ and ψ^{-1} are increasing, it holds that

$$\left((Y_t - \theta Y_t')^+\right)^\eta \le (1-\theta)^\eta \mathbb{E}_{\mathbb{Q}^\gamma}\left[\psi^{-1}\left(\psi\left(|X|^\eta + \int_0^T e^\eta \alpha_s ds\right) + \int_t^T (\alpha_s+\beta_s)ds\right) \middle| \mathcal{F}_t\right].$$

Letting $\theta \to 1$, the thesis follows. □

Proof of Theorem 17. Existence is clear by Theorem 11. The uniqueness for Y is obvious by Proposition 18 and pathwise continuity of Y. We prove uniqueness for Z. Let us consider (Y, Z) and (Y', Z'), being two solutions to Equation (3.5). Apply Itô's formula to $(Y - Y')^2$:

$$(Y_t - Y_t')^2 = \int_t^T 2(Y_s - Y_s')(g(s, Y_s, Z_s) - g(s, Y_s', Z_s')) - |Z_s - Z_s'|^2 ds - \int_t^T 2(Y_s - Y_s')(Z_s - Z_s')dW_s.$$

Since we already know that $Y = Y'$, the previous equation gives

$$\int_0^T |Z_s - Z_s'|^2 ds = 0 \ d\mathbb{P}\text{-a.s.},$$

and the thesis follows. □

4.4 Stability

Stability results play a central role in the theory of BSDEs as well as in their applications involving numerical methods. The analysis of the stability of solutions has been an important topic since the earliest works on BSDEs (e.g., [26, 36]) and continues to attract attention in the recent literature, e.g., in [46]. In this latter contribution, the authors establish general stability results for Lipschitz BSDEs in an enlarged filtration, thereby extending earlier findings on the stability of BSDEs and providing a unified framework that encompasses several approaches to numerical approximation and implementation.

In this subsection, we derive stability results for the BSDE (3.5)–(3.6) under assumptions A) and C). In the following, we employ again the convexity of the driver to obtain an estimation involving $|Y^n - \theta Y|$, and then let $\theta \to 1$ to establish the thesis. This approach has been studied in the context of stability in [13]. However, our proof strategy is different from the strategy used by these authors. Indeed, we construct a proof based on the comparison results obtained in Proposition 18. Before proceeding, we need some refinements concerning the regularity of the z-component of the solution to Equation (3.5).

Proposition 19. *With the same notation as in Theorem 11, considering assumption A), $X \in L_+^{p(2\delta+1)(e^B+1)}$ with $p > 1$ and $B = \|\beta\|_\infty^T \cdot T$, any solution (Y, Z) to Equation (3.5) such that $0 < Y \le Y^h$ verifies for some $C > 0$ depending on $p, \|\alpha\|_\infty^T, \|\beta\|_\infty^T, \|\gamma\|_\infty^T, \delta$ and T:*

$$\mathbb{E}\left[\left(\int_0^T |Z_s|^2 ds\right)^p\right] + \mathbb{E}\left[\left(\int_0^T g(s, Y_s, Z_s) ds\right)^{2p}\right] \le C\mathbb{E}\left[X^{p(2\delta+1)(e^B+1)}\right] < +\infty.$$

Furthermore, if there exists $\varepsilon > 0$ such that $X \ge \varepsilon$ a.s., then it holds that

$$\mathbb{E}\left[\left(\int_0^T \frac{|Z_s|^2}{Y_s} ds\right)^p\right] \le C'\mathbb{E}\left[X^{p(2\delta+1)(e^B+1)}\right] < +\infty,$$

for some $C' > 0$ depending on $p, \|\alpha\|_\infty^T, \|\beta\|_\infty^T, \|\gamma\|_\infty^T, \delta$ and T.

Proof. For brevity, we assume the coefficients α, β, γ to be constant, but the same results are valid when considering bounded random coefficients as in assumption A).

We start by proving the regularity for the z-component. If Y is bounded there is nothing to prove since $Z \in \mathrm{BMO}(\mathbb{P})$ (see Proposition 14) and thus $Z \in \mathcal{M}_T^p$ for any $p \ge 1$. Suppose, instead, that Y is sufficiently large. Proceeding as in Theorem 11, we obtain the analog of Equation (4.1) with $\eta \ge 2$ and $\eta > 2\delta + 1$:

$$\begin{aligned}&\int_0^T \eta\left(\frac{\eta-1}{2} - \delta\right) Y_s^{\eta-2}|Z_s|^2 ds \\ &\le -Y_0^\eta + X^\eta + \int_0^T \left(\eta Y_s^{\eta-1}(\alpha Y_s + \beta Y_s|\ln(Y_s)| + \gamma|Z_s|)\right) ds - \int_0^T \eta Y_s^{\eta-1} Z_s dW_s.\end{aligned}$$

Using Young's inequality, we obtain

$$\begin{aligned}&\int_0^T \eta\left(\frac{\eta-1}{2} - \delta\right) Y_s^{\eta-2}|Z_s|^2 ds \\ &\le C\left(\sup_{t\in[0,T]} Y_t^{\eta+\varepsilon'} + \sup_{t\in[0,T]} Y_t^\eta\right) + \frac{1}{2m}\int_0^T Y_s^{\eta-2}|Z_s|^2 ds - \int_0^T \eta Y_s^{\eta-1} Z_s dW_s,\end{aligned}$$

with $m > 0$, $\varepsilon' > 0$ to be determined later and $C > 0$ depending on $T, \varepsilon', m, \alpha, \beta, \gamma$. In the following, the constant $C > 0$ can vary from line to line and it will not be renamed for ease of exposition. Choosing a suitable $m > 0$, there exists $K > 0$ such that

$$K\int_0^T \eta Y_s^{\eta-2}|Z_s|^2 ds \le C\sup_{t\in[0,T]} Y_t^{\eta+\varepsilon'} - \int_0^T \eta Y_s^{\eta-1} Z_s dW_s.$$

Raising both members to the p-th power and taking the expectation, we obtain

$$K\mathbb{E}\left[\left(\int_0^T \eta Y_s^{\eta-2}|Z_s|^2 ds\right)^p\right] \leq C\left(\mathbb{E}\left[\sup_{t\in[0,T]} Y_t^{p\eta+\varepsilon''}\right] + \mathbb{E}\left[\left|\int_0^T \eta Y_s^{\eta-1} Z_s dW_s\right|^p\right]\right), \tag{4.6}$$

with $\varepsilon'' = p\varepsilon'$. Considering the localization $\tau_n := \inf\{t \in [0,T] : \int_0^t (\eta Y_s^{\eta-1} Z_s)^2 ds \geq n\} \wedge T$ and employing Burkholder-Davis-Gundy's (BDG's) and Young's inequalities, it holds that

$$\mathbb{E}\left[\left|\int_0^{\tau_n} \eta Y_s^{\eta-1} Z_s dW_s\right|^p\right] \leq C\mathbb{E}\left[\left(\int_0^{\tau_n} \eta^2 Y_s^{2(\eta-1)}|Z_s|^2 ds\right)^{p/2}\right]$$

$$\leq C\mathbb{E}\left[\sup_{t\in[0,T]} Y_t^{p\eta} + \frac{1}{2m'}\left(\int_0^{\tau_n} Y_s^{\eta-2}|Z_s|^2 ds\right)^p\right], \tag{4.7}$$

where $C > 0$ is a constant depending on p and m'. Choosing a suitable $m' > 0$ and combining Equations (4.6) and (4.7), we get

$$K\mathbb{E}\left[\left(\int_0^{\tau_n} Y_s^{\eta-2}|Z_s|^2 ds\right)^p\right] \leq C\mathbb{E}\left[\sup_{t\in[0,T]} Y_t^{p\eta+\varepsilon''}\right].$$

Letting $n \to \infty$, we have by Fatou's lemma

$$K\mathbb{E}\left[\left(\int_0^{T} Y_s^{\eta-2}|Z_s|^2 ds\right)^p\right] \leq C\mathbb{E}\left[\sup_{t\in[0,T]} Y_t^{p\eta+\varepsilon''}\right].$$

Proceeding as in Theorem 11, we can select suitable $\eta \geq 2$, $\eta > 2\delta + 1$ and $\varepsilon'' = p\varepsilon' > 0$ such that

$$\mathbb{E}\left[\left(\int_0^{T} Y_s^{\eta-2}|Z_s|^2 ds\right)^p\right] \leq C\mathbb{E}\left[\sup_{t\in[0,T]} Y_t^{p(2\delta+1)(e^B+1)}\right] \leq C\mathbb{E}\left[X^{p(2\delta+1)(e^B+1)}\right],$$

where the last inequality is implied by Proposition 3.2 of [2]. Then, the bound

$$\mathbb{E}\left[\left(\int_0^{T} |Z_s|^2 ds\right)^p\right] \leq C\mathbb{E}\left[X^{p(2\delta+1)(e^B+1)}\right],$$

follows similarly as the implication $Z \in \mathcal{M}_T^2$ in Theorem 11.

Now we prove the regularity of the process $g(\cdot, Y_\cdot, Z_\cdot)$. Rearranging Equation (3.5) and recalling that $g \geq 0$, it holds that

$$\left(\int_0^T |g(s, Y_s, Z_s)| ds\right)^{2p} \leq c_p\left(\sup_{t\in[0,T]} Y_t^{2p} + \left|\int_0^T Z_s dW_s\right|^{2p}\right),$$

with $c_p > 0$ depending on p. Taking the expectation and using BDG's inequality, there exists another constant $c_p > 0$ such that

$$\mathbb{E}\left[\left(\int_0^T |g(s, Y_s, Z_s)| ds\right)^{2p}\right] \leq c_p\mathbb{E}\left[\sup_{t\in[0,T]} Y_t^{2p} + \left(\int_0^T |Z_s|^2 ds\right)^p\right] < +\infty.$$

Now we assume there exists $\varepsilon > 0$ such that $X \geq \varepsilon$ a.s., thus also $Y_t \geq \varepsilon$ $d\mathbb{P} \times dt$-a.s., and the thesis follows by z-regularity. □

Theorem 20. *Utilizing the same notation as in Theorem 17, consider a family of parameters $(X^n, g^n)_{n\in\mathbb{N}}$ and (X, g) with g satisfying assumptions A) and C). Let $\sup_n g^n$ verify A), g^n verify C) for any $n \in \mathbb{N}$, and $(X^n)_{n\in\mathbb{N}}, X$ be strictly positive random variables such that $\sup_{n\in\mathbb{N}} X^n, X \in L^{p(2\delta+1)(e^B+1)}$, with $p > \max\{1, \frac{2}{2\delta+1}\}$. We call (Y, Z) (resp. (Y^n, Z^n)) the unique solution to the BSDE with parameters (X, g) (resp. (X^n, g^n)) such that $0 < Y \le Y^h$ (resp. $0 < Y^n \le Y^h$), with h as in A). Furthermore, assume $X^n \to X$ $d\mathbb{P}$-a.s. and $\int_0^T g^n(t, Y_t, Z_t)dt \to \int_0^T g(t, Y_t, Z_t)dt$ in $L^{p(2\delta+1)(e^B+1)}$. Then, $(Y^n, Z^n) \to (Y, Z)$ in $L^q \times \mathcal{M}_T^{2p}$ for any $q \in [1, p(2\delta + 1)(e^B + 1))$.*

Proof. From Theorem 11, Proposition 19 and the assumptions on $(X^n, g^n)_{n\in\mathbb{N}}$, it follows that

$$\sup_{n\in\mathbb{N}} \mathbb{E}\left[\sup_{t\in[0,T]} (Y_t^n)^{p(2\delta+1)(e^B+1)} + \left(\int_0^T |Z_s^n|^2 ds\right)^p\right] \le K \sup_{n\in\mathbb{N}} \mathbb{E}\left[(X^n)^{p(2\delta+1)(e^B+1)}\right] < +\infty,$$

for some $K > 0$. To establish the thesis, by Vitali's convergence theorem, it is sufficient to show that $\sup_{t\in[0,T]} Y_t^n \to \sup_{t\in[0,T]} Y_t$ in probability. Before proceeding, we show that, if $Y^n \to Y$ in $\mathcal{H}_T^q$ for any $q \in [1, p(2\delta + 1)(e^B + 1))$, then $Z^n \to Z$ in $\mathcal{M}_T^{2p}$. Indeed, by Itô's formula and BDG's inequality, we have

$$\mathbb{E}\left[\left(\int_0^T |Z_s^n - Z_s|^2 ds\right)^p\right] \le C\mathbb{E}\left[|X^n - X|^{2p} + \sup_{t\in[0,T]} |Y_t^n - Y_t|^{2p}\right]$$
$$+ C\mathbb{E}\left[\sup_{t\in[0,T]} |Y_t^n - Y_t|^p \left(\int_0^T |g^n(s, Y_s^n, Z_s^n) - g(s, Y_s, Z_s)|ds\right)^p\right].$$

The first and second terms inside the first expectation in the right-hand member of the previous inequality converge to 0 by assumption, since $2p < p(2\delta + 1)(e^B + 1)$. We only need to analyze the third term. It holds that $\left(\int_0^T |g^n(s, Y_s^n, Z_s^n)|ds\right)^{2p}$ is uniformly bounded in $L^1(\mathcal{F}_T)$; indeed,

$$\left(\int_0^T |g^n(s, Y_s^n, Z_s^n) - g(s, Y_s, Z_s)|ds\right)^{2p}$$
$$\le C\left[\left(\int_0^T |g^n(s, Y_s^n, Z_s^n)|ds\right)^{2p} + \left(\int_0^T |g(s, Y_s, Z_s)|ds\right)^{2p}\right],$$

thus Proposition 19 and integrability assumptions on $(X^n)_{n\in\mathbb{N}}$ give

$$\sup_{n\in\mathbb{N}} \mathbb{E}\left[\left(\int_0^T |g^n(s, Y_s^n, Z_s^n)|ds\right)^{2p}\right] \le C \sup_{n\in\mathbb{N}} \mathbb{E}\left[(X^n)^{p(2\delta+1)(e^B+1)}\right] < +\infty,$$

and similarly for $(\int_0^T |g(s, Y_s, Z_s)|ds)^{2p}$. Thus, Hölder's inequality yields

$$\mathbb{E}\left[\sup_{t\in[0,T]} |Y_t^n - Y_t|^p \left(\int_0^T |g^n(s, Y_s^n, Z_s^n) - g(s, Y_s, Z_s)|ds\right)^p\right]$$
$$\le \mathbb{E}\left[\sup_{t\in[0,T]} |Y_t^n - Y_t|^{2p}\right]^{\frac{1}{2}} \mathbb{E}\left[\left(\int_0^T |g^n(s, Y_s^n, Z_s^n) - g(s, Y_s, Z_s)|ds\right)^{2p}\right]^{\frac{1}{2}}$$
$$\le K\mathbb{E}\left[\sup_{t\in[0,T]} |Y_t^n - Y_t|^{2p}\right]^{\frac{1}{2}} \xrightarrow{n\to\infty} 0.$$

Hence, we have proved $Z^n \to Z$ in $\mathcal{M}_T^{2p}$. Now, we show that $\sup_{t\in[0,T]} Y_t^n \to \sup_{t\in[0,T]} Y_t$ in L^η for some $\eta > 1$, establishing also the convergence in probability. We proceed as in Proposition 18, estimating the difference $Y^n - \theta Y$. As before, we define $P := \frac{Y^n - \theta Y}{1-\theta}$ and $V := \frac{Z^n - \theta Z}{1-\theta}$. It holds that

$$P_t = P_T + \int_t^T G(s, P_s, V_s)ds - \int_t^T V_s dW_s, \tag{4.8}$$

where

$$\begin{aligned} G(\omega,t,y,z) := &\frac{1}{1-\theta}\left[g^n(\omega,t,(1-\theta)y+\theta Y_t(\omega),(1-\theta)z+\theta Z_t(\omega)) - \theta g^n(\omega,t,Y_t(\omega),Z_t(\omega))\right] \\ &+ \frac{\theta}{1-\theta}\left[g^n(\omega,t,Y_t(\omega),Z_t(\omega)) - g(\omega,t,Y_t(\omega),Z_t(\omega))\right], \end{aligned}$$

if $((1-\theta)y + \theta Y_t(\omega), (1-\theta)z + \theta Z_t(\omega)) \in \mathbb{R}_+ \times \mathbb{R}^n$ and $G(\omega, t, y, z) := 0$ otherwise. Convexity of g^n yields

$$\mathbb{I}_{\{y>0\}} G(t,y,z) \le \mathbb{I}_{\{y>0\}} \left(g^n(t,|y|,z) + |\delta^n_\theta g(t)|\right),$$

where $\delta^n_\theta g(\cdot) := \frac{\theta}{1-\theta}\left[g^n(\cdot, Y_\cdot, Z_\cdot) - g(\cdot, Y_\cdot, Z_\cdot)\right]$ and with the convention that the right-hand member of the previous inequality is zero when $y = 0$. Following the same steps as in Proposition 18 and with the same notation of this proposition, we obtain the analog of Equation (4.5):

$$\begin{aligned} &(P_t^+)^\eta \le (P_T^+)^\eta \\ &+ \int_t^T \eta(P_s^+)^\eta(\alpha_s + \beta_s|\ln(P_s^+)|) + \eta\mathbb{I}_{\{P_s>0\}}(P_s^+)^{\eta-1}\gamma_s\cdot V_s + \eta\mathbb{I}_{\{P_s>0\}}(P_s^+)^{\eta-1}|\delta^n_\theta g(s)|ds \\ &- \int_t^T \eta\mathbb{I}_{\{P_s>0\}}(P_s)^{\eta-1}V_s dW_s \\ &\le (P_T^+)^\eta + \sup_{t\in[0,T]}(P_t^+)^{\eta-1}\int_0^T \eta|\delta^n_\theta g(s)|ds \\ &+ \int_t^T \eta(P_s^+)^\eta(\alpha_s + \beta_s|\ln(P_s^+)|) + \eta\mathbb{I}_{\{P_s>0\}}(P_s^+)^{\eta-1}\gamma_s\cdot V_s\, ds - \int_t^T \eta\mathbb{I}_{\{P_s>0\}}(P_s)^{\eta-1}V_s dW_s, \end{aligned}$$

where $\eta = \max\{2, 2\delta+1\}$. By integrability assumptions on $\delta^n_\theta g$ and by the regularity of Y^n and Y, it is easy to check that

$$\Gamma^n_T := \sup_{t\in[0,T]}(P_t^+)^{\eta-1}\int_0^T |\delta^n_\theta g(s)|ds \le C\left(\sup_{t\in[0,T]}(P_t^+)^\eta + \left(\int_0^T |\delta^n_\theta g(s)|ds\right)^\eta\right)$$

verifies the hypotheses of Lemma 16, since $\eta = \max\{2, 2\delta+1\}$ and $p > \max\{1, \frac{2}{2\delta+1}\}$. Thus, we can use Lemma 16 as done in Proposition 18, obtaining

$$(P_t^+)^\eta \le \mathbb{E}_{\mathbb{Q}^\gamma}\left[\psi^{-1}\left(\psi\left((P_T^+)^\eta + \Gamma^n_T + \int_0^T e^\eta\alpha_s ds\right) + \int_0^T(\alpha_s+\beta_s)ds\right)\middle|\mathcal{F}_t\right].$$

Recalling the inequality $(y-y')^+ \le (y-\theta y')^+$ for any $y, y' > 0$ and $\theta \in [0,1]$, the definitions of P^+ and ψ and using Doob's inequality for $p' > 1$ small enough, it holds that

$$\begin{aligned} &\mathbb{E}_{\mathbb{Q}^\gamma}\left[\left(\sup_{t\in[0,T]}(Y_t^n - Y_t)^+\right)^\eta\right] \\ &\le C_{p'}(1-\theta)^\eta\mathbb{E}_{\mathbb{Q}^\gamma}\left[\left(\psi^{-1}\left(\psi\left((P_T^+)^\eta + \Gamma^n_T + \int_0^T e^\eta\alpha_s ds\right) + \int_0^T(\alpha_s+\beta_s)ds\right)\right)^{p'}\right] \\ &\le C_{p'}(1-\theta)^\eta\mathbb{E}_{\mathbb{Q}^\gamma}\left[(P_T^+)^{p'\eta e^B} + (\Gamma^n_T)^{p'e^B} + \left(\int_0^T e^\eta\alpha_s ds\right)^{p'e^B}\right]. \end{aligned} \tag{4.9}$$

For brevity and w.l.o.g. we can assume $\mathbb{Q}^\gamma = \mathbb{P}$, i.e., $\gamma \equiv 0$. For each fixed $\theta \in [0,1]$, the integrability assumption on X^n yields

$$\mathbb{E}\left[(P_T^+)^{p'\eta e^B}\right] = \mathbb{E}\left[\left(\frac{(X^n-\theta X)^+}{1-\theta}\right)^{p'\eta e^B}\right] \xrightarrow{n\to\infty} \mathbb{E}\left[X^{p'\eta e^B}\right].$$

Furthermore, it holds that

$$\begin{aligned}\mathbb{E}\left[(\Gamma_T^n)^{p'e^B}\right] &\le \mathbb{E}\left[\sup_{t\in[0,T]}(P_t^+)^{p'\eta e^B}\right]^{\frac{\eta-1}{\eta}} \mathbb{E}\left[\left(\int_0^T |\delta_\theta^n g(s)|ds\right)^{p'\eta e^B}\right]^{\frac{1}{\eta}}\\ &\le C\mathbb{E}\left[\left(\int_0^T |\delta_\theta^n g(s)|ds\right)^{p'\eta e^B}\right]^{\frac{1}{\eta}} \xrightarrow{n\to\infty} 0.\end{aligned}$$

Here, in the first inequality we have used Hölder's inequality with $l = \frac{\eta}{\eta-1}$ and $m = \eta$ as conjugate exponents, the second inequality follows from $L^{p'\eta e^B}$-boundedness of $\sup_{t\in[0,T]} P_t^+$, while the convergence is due to the integrability assumptions on $\delta_\theta^n g$. Thus, letting $n\to\infty$ in Equation (4.9), we obtain

$$\lim_{n\to\infty} \mathbb{E}\left[\sup_{n\in\mathbb{N}}((Y_t^n - Y_t)^+)^\eta\right] \le (1-\theta)^\eta \mathbb{E}\left[X^{p'\eta e^B} + \left(\int_0^T e^\eta \alpha_s ds\right)^{p'e^B}\right].$$

The left-hand member in the previous inequality does not depend on θ, thus we can let $\theta \to 1$, obtaining $\lim_{n\to\infty}\mathbb{E}\left[\sup_{t\in[0,T]}((Y_t^n - Y_t)^+)^\eta\right] = 0$. Similarly, we can evaluate the difference $Y - \theta Y^n$, yielding the analog of Equation (4.8) with $P' := \frac{Y-\theta Y^n}{1-\theta}$, $V' := \frac{Z-\theta Z^n}{1-\theta}$ and

$$\begin{aligned}G'(t,y,z) :=& \frac{1}{1-\theta}\left[g^n(\omega,t,(1-\theta)y+\theta Y_t^n(\omega),(1-\theta)z+\theta Z_t^n(\omega)) - \theta g^n(\omega,t,Y_t^n(\omega),Z_t^n(\omega))\right]\\ &+ \frac{1}{1-\theta}\left[g(\omega,t,Y_t(\omega),Z_t(\omega)) - g^n(\omega,t,Y_t(\omega),Z_t(\omega))\right],\end{aligned}$$

if $((1-\theta)y + \theta Y_t(\omega), (1-\theta)z + \theta Z_t(\omega)) \in \mathbb{R}_+ \times \mathbb{R}^n$ and $G(\omega,t,y,z) := 0$ otherwise. Once again, convexity of g^n yields

$$\mathbb{I}_{\{y>0\}}G'(t,y,z) \le \mathbb{I}_{\{y>0\}}\left(g^n(t,y,z) + |\delta_\theta^n g'(t)|\right),$$

with $\delta_\theta^n g'(\cdot) := \frac{1}{1-\theta}\left[g(\cdot,Y_\cdot,Z_\cdot) - g^n(\cdot,Y_\cdot,Z_\cdot)\right]$. Performing exactly the same algebra as before, we obtain

$$\lim_{n\to\infty}\mathbb{E}\left[\sup_{t\in[0,T]}\left((Y_t - Y_t^n)^+\right)^\eta\right] \le (1-\theta)^\eta \mathbb{E}\left[X^{p'\eta e^B} + \left(\int_0^T e^\eta \alpha_s ds\right)^{p'e^B}\right],$$

which gives $\lim_{n\to\infty}\mathbb{E}\left[\sup_{t\in[0,T]}((Y_t - Y_t^n)^+)^\eta\right] = 0$. Thus, we have proved

$$\lim_{n\to\infty}\mathbb{E}\left[\sup_{t\in[0,T]}|Y_t - Y_t^n|^\eta\right] = 0,$$

and the first thesis follows. □

The preceding stability results are provided under general hypotheses. However, verifying the condition $\int_0^T g^n(t,Y_t,Z_t)dt \to \int_0^T g(t,Y_t,Z_t)dt$ in $L^{p(2\delta+1)(e^B+1)}$ might be challenging in practical applications. If $g^n \equiv g$ for any $n\in\mathbb{N}$, then the above integrability condition is not needed. Indeed, stability of the solution holds automatically, as is clear from the proof of Theorem 20, since in this circumstance it results that $\delta g_\theta^n \equiv 0$ for any $\theta\in[0,1]$ and $n\in\mathbb{N}$. In addition, in the following corollary, we provide some other sufficient conditions under which stability holds.

Corollary 21. *With the same notation and hypotheses as in Theorem 20, let* $X^n \to X$ $d\mathbb{P}$*-a.s. and* $\sup_n X^n, X \in L^p$ *for any* $p \geq 1$*. Assume that,* $d\mathbb{P} \times dt$*-a.s. for any* $(y, z) \in \mathbb{R}_+ \times \mathbb{R}^n$*,* $g^n(t, y, z) \to g(t, y, z)$ *and that there exists* $\varepsilon > 0$ *such that* $X \geq \varepsilon$ $d\mathbb{P}$*-a.s. Then,* $(Y^n, Z^n) \to (Y, Z)$ *in* $\mathcal{H}^p_T \times \mathcal{M}^p_T$ *for any* $p \geq 1$*.*

Remark 22. *The additional assumption* $\sup_n X_n, X \in L^p$ *for any* $p \geq 1$ *in Corollary 21 is not unnatural. Indeed, [13] assumed the finiteness of all exponential moments for the terminal conditions of quadratic BSDEs (see Proposition 7 of [13]), while Proposition 8 suggests that exponential regularity for the terminal conditions of quadratic BSDEs should naturally lead to* L^p *regularity for the terminal conditions when the drivers exhibit an LN-Q growth rate. Similarly, the hypothesis* $X \geq \varepsilon$ *in Corollary 21 appears to be pivotal in case one seeks further regularities, as shown, for example, in [6], and, within the framework of return risk measures, in [9] and [38].*

5 Two-Driver BSDEs

With the existence, regularity, uniqueness and stability results for ordinary LN-Q BSDEs of Section 4 at hand, we are now equipped to examine general two-driver BSDEs, embedding Equation (3.4) and given by

$$Y_t = X + \int_t^T g_1(s, Y_s, Z_s)ds - \int_t^T g_2(s, Y_s, Z_s)dW_s. \tag{5.1}$$

Here, g_1 verifies the growth rate

$$0 \leq g_1(t, y, z) \leq y(\alpha_t + \beta_t|\ln(y)| + \gamma_t|z| + \delta|z|^2) =: h_1(t, y, z), \quad \forall (y, z) \in \mathbb{R}_+ \times \mathbb{R}^n, \tag{5.2}$$

with α, β, γ non-negative and predictable stochastic processes, and $\delta > 0$. Furthermore, the driver $g_2 : [0, T] \times \Omega \times \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies $\mathcal{P} \times \mathcal{B}(\mathbb{R}_+) \times \mathcal{B}(\mathbb{R}^n)/\mathcal{B}(\mathbb{R}^n)$-measurability. We first analyze under which additional assumptions on g_2 it is possible to reduce Equation (5.1) to Equation (3.5). A sufficient growth rate on g_2 is provided in the following proposition. We start by defining a general notion of a solution to a two-driver BSDE.

Definition 23. *The couple* (Y, Z) *is a solution to Equation (5.1) if it satisfies Equation (5.1) in the Itô sense,* Y *is a continuous and predictable process, and* $Z \in \mathcal{L}^2_T$ *is a predictable process. Furthermore,* (Y, Z) *is required to satisfy* $\int_0^T g_1(s, Y_s, Z_s)ds < +\infty$ *and* $\int_0^T |g_2(s, Y_s, Z_s)|^2 ds < +\infty$ $d\mathbb{P}$*-a.s.*

Proposition 24. *Let* $g_1 : [0, T] \times \Omega \times \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ *be a* $\mathcal{P} \times \mathcal{B}(\mathbb{R}_+) \times \mathcal{B}(\mathbb{R}^n)/\mathcal{B}(\mathbb{R}_+)$*-measurable function satisfying Equation (5.2) and let* $g_2 : [0, T] \times \Omega \times \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ *be a* $\mathcal{P} \times \mathcal{B}(\mathbb{R}_+) \times \mathcal{B}(\mathbb{R}^n)/\mathcal{B}(\mathbb{R}^n)$*-measurable function.*

If there exists $K > 0$ *such that* $d\mathbb{P} \times dt$*-a.s. for any* $(y, z) \in \mathbb{R}_+ \times \mathbb{R}^n$ *it holds that* $|g_2(t, y, z)| \geq Ky|z|$ *with* g_2 *being injective and continuous in* z*, then Equation (5.1) admits a positive solution in the sense of Definition 23 when the ordinary BSDE (3.5) with LN-Q growth rate does.*

Proof. First, we note that the mapping $z \mapsto g_2(\omega, t, y, z)$ is a bijection from $\mathbb{R}^n$ into itself. The injective continuity of g_2 w.r.t. z and the condition $\lim_{|z| \to \infty} |g_2(\omega, t, y, z)| = +\infty$, established by the inequality $|g_2(t, \omega, y, z)| \geq Ky|z|$, ensure the surjectivity of g_2 as a consequence of the proof of Theorem 59 (V) in [51]. Hence, the bijectivity of g_2 with respect to z enables us to find the inverse of the function $z \mapsto g_2(\omega, t, y, z)$ $d\mathbb{P} \times dt$-a.s., for any $y \in \mathbb{R}_+$. In particular, for any $(\omega, t, y) \in \Omega \times [0, T] \times \mathbb{R}_+$, there exists a unique function $v \mapsto f^{\omega,t,y}(v)$ such that $g_2(\omega, t, y, f^{\omega,t,y}(v)) = v = f^{\omega,t,y}(g_2(\omega, t, y, v))$ for any $v \in \mathbb{R}^n$. With slight abuse of notation, we denote this inverse map by $(\omega, t, y, v) \mapsto g_2^{-1}(\omega, t, y, v) =: f^{\omega,t,y}(v)$. Furthermore, the growth condition on g_2 implies: $|z| \leq \frac{1}{K}|g_2(t, y, z)|/y$, which can be translated into: $|g_2^{-1}(t, y, v)| \leq$

$\frac{1}{K}|v|/y$, for any $(y,v) \in \mathbb{R}_+ \times \mathbb{R}^n$, $d\mathbb{P} \times dt$-a.s. In addition, employing the substitution $V_\cdot := g_2(\cdot, Y_\cdot, Z_\cdot)$, Equation (5.1) becomes

$$Y_t = X + \int_t^T g_1(s, Y_s, g_2^{-1}(s, Y_s, V_s))ds - \int_t^T V_s dW_s. \tag{5.3}$$

Now we check that $g(t,y,v) := g_1(t,y,g_2^{-1}(t,y,v))$ verifies the LN-Q growth rate:

$$\begin{aligned} 0 \leq g(t,y,v) = g_1(t,y,g_2^{-1}(t,y,v)) &\leq \alpha_t y + \beta_t y|\ln(y)| + \gamma_t y|g_2^{-1}(t,y,v)| + \delta y|g_2^{-1}(t,y,v)|^2 \\ &\leq \alpha_t y + \beta_t y|\ln(y)| + \frac{\gamma_t|v|}{K} + \frac{\delta|v|^2}{K^2 y}. \end{aligned}$$

It only remains to check that g is a $\mathcal{P} \times \mathcal{B}(\mathbb{R}_+) \times \mathcal{B}(\mathbb{R}^n)/\mathcal{B}(\mathbb{R}_+)$-measurable function. To show this, it is sufficient to prove that $(\omega,t,y,v) \mapsto (\omega,t,y,g_2^{-1}(t,y,v))$ is a $\mathcal{P} \times \mathcal{B}(\mathbb{R}_+) \times \mathcal{B}(\mathbb{R}^n)$-measurable function, since the composition of measurable functions preserves this property. Adapting the argument provided in the proof of Theorem 4.1 in [47], we can consider w.l.o.g. the space $\Omega = C^0([0,T],\mathbb{R}^n)$ with the canonical Brownian Motion $W_t(\omega) = \omega(t)$. Then, the function $G(\omega,t,y,z) := (\omega,t,y,g_2(\omega,t,y,z))$ is a bijection of $\Omega \times [0,T] \times \mathbb{R}_+ \times \mathbb{R}^n$ into itself. Consequently, its inverse $G^{-1}(\omega,t,y,v) = (\omega,t,y,g_2^{-1}(\omega,t,y,v))$ is $\mathcal{P} \times \mathcal{B}(\mathbb{R}_+) \times \mathcal{B}(\mathbb{R}^n)$-measurable. This measurability is ensured by the completeness and separability of the metric space Ω (see, e.g., Theorem 10.5 in [28]). Thus, if (Y,V) is a solution to Equation (5.3), then $(Y_\cdot, Z_\cdot) := (Y_\cdot, g_2^{-1}(\cdot, Y_\cdot, V_\cdot))$ is a solution to Equation (5.1). □

Remark 25. *Proposition 24 entails that if we are able to find a solution to Equation (5.3), then also a solution to Equation (5.1) exists. Nevertheless, the regularity of the second component of the solution, $V_\cdot$, still depends on the properties of g_2. Importantly, this means that, in the following, once we handle a solution with a certain regularity to Equation (5.3), we will need to explore the regularity of the second component of the solution to Equation (5.1) as a separate problem.*

In the following, we prove results about existence and uniqueness for general two-driver BSDEs. Informed by the analysis above, we require the following assumptions:

(G1) $g_1 : [0,T] \times \Omega \times \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ is a $\mathcal{P} \times \mathcal{B}(\mathbb{R}_+) \times \mathcal{B}(\mathbb{R}^n)/\mathcal{B}(\mathbb{R}_+)$-measurable function verifying the growth rate (5.2) and is continuous in (y,z).

(G2) $g_2 : [0,T] \times \Omega \times \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is a $\mathcal{P} \times \mathcal{B}(\mathbb{R}_+) \times \mathcal{B}(\mathbb{R}^n)/\mathcal{B}(\mathbb{R}^n)$-measurable function such that $|g_2(t,y,z)| \geq Ky|z| =: h_2(t,y,z)$, for some $K > 0$. Furthermore, g_2 is injective and continuous in z. Moreover, its inverse[4] with respect to z, denoted with slight abuse of notation as $g_2^{-1}(t,\omega,y,v)$ and defined for any $(t,\omega,y,v) \in \Omega \times [0,T] \times \mathbb{R}_+ \times \mathbb{R}^n$, is continuous in (y,z), $d\mathbb{P} \times dt$-a.s.

(G3) Let $p > 1$. Let α, β, γ be the coefficients appearing in Equation (5.2), with $\delta > 0$, and let K be as in (G2) such that

$$\mathbb{E}\left[(1 + X^{\frac{2\delta}{K^2}+1})^{p(e^B+1)} \exp\left(p(e^B+1)\left((A+B) + \frac{1}{4k}\int_0^T (\gamma_t/K)^2 dt\right)\right)\right] < +\infty.$$

Here, $k \in (0, \frac{1 \wedge (p-1)}{2})$, $A := \int_0^T \alpha_t dt$, and $B := \int_0^T \beta_t dt$. Furthermore, there exists $q' > 0$ such that $\mathbb{E}[\int_0^T e^{q'\gamma_t} dt] < +\infty$ and $\alpha, \beta \in \mathcal{H}_T^q$, with $\frac{1}{q} + \frac{1}{p} = 1$.

[4]See the proof of Proposition 24 for the formal definition of the z-inverse of g_2.

Proposition 26. *Assume (G1), (G2) and (G3). Then, Equation (5.1) admits at least one solution (Y, Z) such that $0 < Y \leq Y^{h_1,h_2}$, where Y^{h_1,h_2} is a solution to Equation (5.1) with parameters (X, h_1, h_2). Here, h_1 and h_2 are defined as in Equation (5.2) and assumption (G2), respectively. In addition, we have the following regularities:*

$$\mathbb{E}\left[\sup_{t\in[0,T]} Y_t^{p\left(\frac{2\delta}{K^2}+1\right)(e^B+1)}\right] < +\infty \quad \text{and} \quad Z \in \mathcal{L}_T^2.$$

Furthermore, if there exists $\varepsilon > 0$ such that $X \geq \varepsilon$, then $Z \in \mathcal{M}_T^2$.

Proof. By Proposition 24, the two-driver BSDE (5.1) admits a positive solution such that $0 < Y \leq Y^{h_1,h_2}$ as soon as the following ordinary BSDE does:

$$Y_t = X + \int_t^T g(s, Y_s, V_s)ds - \int_t^T V_s dW_s. \tag{5.4}$$

Here, $V_\cdot := g_2(\cdot, Y_\cdot, Z_\cdot)$ and $g(t, y, z) := g_1(t, y, g_2^{-1}(t, y, z))$, resulting in $0 \leq g(t, y, z) \leq \alpha_t y + \beta_t y|\ln(y)| + \frac{\gamma_t}{K}|z| + \frac{\delta}{K^2}\frac{|z|^2}{y}$. Furthermore, g is continuous in (y, z) as it is a composition of continuous functions. Hence, the ordinary BSDE (5.4) admits a solution (Y, V) by Theorem 11, with

$$\mathbb{E}\left[\sup_{t\in[0,T]} Y_t^{p\left(\frac{2\delta}{K^2}+1\right)(e^B+1)}\right] < +\infty, \quad V \in \mathcal{M}_T^2,$$

and $0 < Y \leq Y^h$, where $h(t, y, v) := h_1(t, y, h_2^{-1}(t, y, v))$. Thus, we have proved existence of the solution to the two-driver BSDE with parameters (X, g_1, g_2). Note that the y-component of this solution has the desired regularity. Clearly, $Z \in \mathcal{L}_T^2$.

It remains to prove that $Z \in \mathcal{M}_T^2$ under the hypothesis $X \geq \varepsilon$ for some $\varepsilon > 0$. We proceed in two steps.

Step 1: For any $t \in [0, T]$, it holds that $Y_t \geq \varepsilon$ $d\mathbb{P}$-a.s. Indeed, for the conditional expectation in the two-driver BSDE, after a suitable localization $(\tau_n)_{n\in\mathbb{N}}$, it holds that

$$Y_t = \mathbb{E}\left[X + \int_t^{\tau_n} g_1(s, Y_s, Z_s)\middle|\mathcal{F}_t\right] \geq \mathbb{E}\left[X|\mathcal{F}_t\right] \geq \varepsilon,$$

where we have employed the positivity of g_1.

Step 2: Apply Itô's formula to the dynamics of Y, using the function $f(x) = x^\eta$, with $\eta > 0$ to be determined later. We note that Itô's formula can be applied since $Y > 0$ and f is twice differentiable on $(0, +\infty)$ for any $\eta > 0$. We have

$$\begin{aligned}
Y_t^\eta &= X^\eta + \int_t^T \eta Y_s^{\eta-1} g_1(s,Y_s,Z_s) - \frac{\eta(\eta-1)}{2} Y_s^{\eta-2}|g_2(s,Y_s,Z_s)|^2 ds - \int_t^T \eta Y_s^{\eta-1} g_2(s,Y_s,Z_s)dW_s \\
&\leq X^\eta + \int_t^T \eta Y_s^{\eta-1}(\alpha_s Y_s + \beta_s Y_s|\ln(Y_s)| + \gamma_s Y_s|Z_s| + \delta Y_s|Z_s|^2) - \frac{K^2\eta(\eta-1)}{2} Y_s^{\eta-2} Y_s^2 |Z_s|^2 ds \\
&\quad - \int_t^T \eta Y_s^{\eta-1} g_2(s, Y_s, Z_s)dW_s \\
&= X^\eta + \int_t^T \eta Y_s^\eta(\alpha_s + \beta_s|\ln(Y_s)| + \gamma_s|Z_s|)ds \\
&\quad + \int_t^T \eta\left(\delta - \frac{K^2(\eta-1)}{2}\right) Y_s^\eta |Z_s|^2 ds - \int_t^T \eta Y_s^{\eta-1} g_2(s, Y_s, Z_s)dW_s,
\end{aligned}$$

because, by (G2), $|g_2(t,y,z)| \geq Ky|z|$. Taking $\eta > \frac{2\delta}{K^2}+1$, $t=0$ and rearranging, we obtain

$$\begin{aligned}&\int_0^T \eta\left(\frac{K^2(\eta-1)}{2}-\delta\right)Y_s^\eta|Z_s|^2ds\\&\leq -Y_0^\eta + X^\eta + \int_0^T \eta Y_s^\eta(\alpha_s+\beta_s|\ln(Y_s)|+\gamma_s|Z_s|)ds - \int_0^T \eta Y_s^{\eta-1}g_2(s,Y_s,Z_s)dW_s. \qquad (5.5)\end{aligned}$$

Since $Y \geq \varepsilon$ and $\eta > 0$, there exists a positive constant $K' := \varepsilon^\eta > 0$ such that

$$K'\int_0^T \eta\left(\delta-\frac{K^2(\eta-1)}{2}\right)|Z_s|^2ds \leq \int_0^T \eta\left(\delta-\frac{K^2(\eta-1)}{2}\right)Y_s^\eta|Z_s|^2ds.$$

We recall that $\mathbb{E}[\int_0^T \gamma_t^m dt] < +\infty$ for any $m \in [1,+\infty)$ since γ has a finite q'-exponential moment. After a suitable localization, we can take the expectation in Equation (5.5). By applying Young's inequality it holds that

$$\begin{aligned}&K'\mathbb{E}\left[\int_0^{\tau_n}\eta\left(\frac{K^2(\eta-1)}{2}-\delta\right)|Z_s|^2ds\right]\\&\leq \mathbb{E}\left[-Y_0^\eta+X^\eta+\int_0^T\frac{(\eta\alpha_s)^q}{q}+\frac{Y_s^{p\eta}}{p}+\frac{(\eta\beta_s)^q}{q}+Y_s^{p\eta}|\ln^p(Y_s)|+\frac{(\eta\gamma_sY_s^\eta)^2}{2\varepsilon}+\frac{\varepsilon}{2}|Z_s|^2ds\right].\end{aligned}$$

Choosing $\frac{\varepsilon}{2} < K'\eta(\frac{K(\eta-1)}{2}-\delta)$, we obtain (for some $C>0$)

$$\begin{aligned}&K''\mathbb{E}\left[\int_0^{\tau_n}|Z_s|^2ds\right]\\&\leq \mathbb{E}\left[-Y_0^\eta+X^\eta+\int_0^T(\eta\alpha_s)^q/q+Y_s^{p\eta}/p+(\eta\beta_s)^q/q+Y_s^{p\eta}|\ln^p(Y_s)|+\frac{(\eta\gamma_s)^{2m}}{2\varepsilon m}+\frac{Y_s^{2\eta l}}{2\varepsilon l}ds\right]\\&\leq C\left(1+\mathbb{E}\left[\sup_{t\in[0,T]}Y_t^{\eta p+\varepsilon'}+\sup_{t\in[0,T]}Y_t^{2\eta l}\right]\right),\end{aligned}$$

with $K'' = K'\eta(\frac{K(\eta-1)}{2}-\delta)-\frac{\varepsilon}{2} > 0$, $1 < l < p$ and $\frac{1}{m}+\frac{1}{l} = 1$. Here, the first inequality follows from Young's inequality, and $\varepsilon' > 0$ has to be determined. Since we have $\mathbb{E}[\sup_{t\in[0,T]} Y_t^{p(\frac{2\delta}{K^2}+1)(e^B+1)}] < +\infty$, where $e^B+1 \geq 2$ $d\mathbb{P}$-a.s., we can obtain the integrability of the left-hand member of the previous inequality by taking $\eta \leq \frac{p}{l}(\frac{2\delta}{K^2}+1)$. In addition, we also need $\eta p+\varepsilon' = 2p(\frac{2\delta}{K^2}+1)$ and choosing η such that $0 < \varepsilon' = 2p(\frac{2\delta}{K^2}+1)-\eta p$, we get the further condition $\eta < 2(\frac{2\delta}{K^2}+1)$. In sum, we have the following conditions on η:

$$\begin{cases}\eta < 2(\frac{2\delta}{K^2}+1)\\ \eta > 0,\\ \eta \leq \frac{p}{l}(\frac{2\delta}{K^2}+1)\\ \eta > \frac{2\delta}{K^2}+1.\end{cases}$$

It is possible to check that for any value of $\delta \in \mathbb{R}_+$ there exists $\eta > 0$ verifying all the above conditions, as $\frac{p}{l} > 1$. Applying Fatou's lemma, we obtain $\mathbb{E}[\int_0^T |Z_s|^2ds] < +\infty$. □

Remark 27. *If $\gamma \equiv 0$ and (G3) is replaced by the weaker assumption*

$$\mathbb{E}\left[(1+X^{\frac{2\delta}{K^2}+1})^{e^B}\exp\left(e^B(A+B)\right)\right] < +\infty,$$

the existence of a positive solution to Equation (5.1) is still guaranteed, with the regularity

$$0 < \sup_{t\in[0,T]}\mathbb{E}\left[Y_t^{\frac{(2\delta+1)e^B}{K^2}}\right] < +\infty,\ Z \in \mathcal{L}_T^2.$$

This follows from the proof above, using Proposition 10 (i).

In what follows, we provide sufficient conditions to ensure the uniqueness of the solution to Equation (5.1). We will rely on Proposition 24, and on Theorem 17 for the corresponding one-driver BSDE. We will show that under suitable hypotheses on the composition between g_1 and g_2, we can obtain a general result of uniqueness ensuring, for instance, that Equation (3.4) admits a unique solution, assuming unbounded terminal conditions and without requiring a monotonicity condition w.r.t. y for the driver $\tilde{f}$. In addition, we show that uniqueness results hold for a larger class of drivers than those used in Equation (3.4). We require the following two assumptions.

A') g_1 is a $\mathcal{P} \times \mathcal{B}(\mathbb{R}_+) \times \mathcal{B}(\mathbb{R}^n)/\mathcal{B}(\mathbb{R}_+)$-measurable function such that

$$0 \le g_1(t,y,z) \le \alpha_t y + \beta_t y|\ln(y)| + \delta y|z|^2, \quad \forall (y,z) \in \mathbb{R}_+ \times \mathbb{R}^n,$$

where α, β are bounded and non-negative stochastic processes, and $\delta > 0$.

C') Given g_1 as in A') and g_2 as in (G2), the function $(\omega,t,y,v) \mapsto g_1(\omega,t,y,g_2^{-1}(t,\omega,y,v))$ is convex in (y,v) $d\mathbb{P} \times dt$-a.s.

Clearly, we can also add the term "$y\gamma_t \cdot z$" in assumption A'), with $\gamma \in \mathcal{H}_T^\infty$, without altering any of the following results.

Proposition 28. *Assume that the drivers g_1 and g_2 verify assumptions A') and C'). Let $X \in L_+^{p(\frac{2\delta}{K^2}+1)(e^B+1)}$, where $p > \max\{\frac{2K^2}{2\delta+K^2}, 1\}$ and $B = (\|\alpha\|_\infty^T + \|\beta\|_\infty^T)T$. Then, the two-driver BSDE (5.1) with parameters (X, g_1, g_2) admits a unique solution*

$$(Y,Z) \in \mathcal{H}_T^{p(\frac{2\delta}{K^2}+1)(e^B+1)} \times \mathcal{L}_T^2$$

among the class of solutions such that $0 < Y \le Y^{h_1,h_2}$, where Y^{h_1,h_2} is the (maximal) solution corresponding to the parameters (X, h_1, h_2) with h_1 and h_2 as defined in Theorem 26.[5]

We can also state a comparison principle for two-driver BSDEs. The proof of the following comparison result is omitted since it can be obtained by similar arguments as those in the proof of Proposition 28.

Proposition 29. *With the same notation as in Proposition 28, consider drivers g_1 and g_2 (resp. g_1' and g_2') verifying assumptions A'), C') and $X, X' \in L_+^{p(2\delta+1)(e^B+1)}$ with $X \le X'$. If $(Y,Z) \in \mathcal{H}_T^{p(2\delta+1)(e^B+1)} \times \mathcal{L}_T^2$ is a solution to the two-driver BSDE with parameters (X, g_1, g_2), $(Y',Z') \in \mathcal{H}_T^{p(2\delta+1)(e^B+1)} \times \mathcal{L}_T^2$ is a solution to the BSDE with parameters (X', g_1', g_2') such that $0 < Y, Y' \le Y^{h_1,h_2}$ and the drivers verify*

$$g_1(t,Y_t',g_2^{-1}(t,Y_t',Z_t')) \le g_1'(t,Y_t',(g_2')^{-1}(t,Y_t',Z_t')) \ d\mathbb{P} \times dt\text{-a.s.}$$
$$(\text{resp. } g_1(t,Y_t,g_2^{-1}(t,Y_t,Z_t)) \le g_1'(t,Y_t,(g_2')^{-1}(t,Y_t,Z_t)) \ d\mathbb{P} \times dt\text{-a.s.}),$$

then, $d\mathbb{P}$-a.s., $Y_t \le Y_t'$ for any $t \in [0,T]$.

Corollary 30. *(i) Consider Equation (3.4). Assume that*

$$0 \le \tilde{f}(t,y,z) \le \alpha_t + \beta_t|\ln(y)| + \delta|z|^2 =: \tilde{h}(t,y,z),$$

where the coefficients α, β and the terminal condition X verify (G3) with $K = 1$ and $\gamma \equiv 0$. Then, there exists at least one solution (Y,Z) to Equation (3.4) such that $0 < Y \le Y^{\tilde{h}}$, where

[5] Here, "unique solution" means that if (Y,Z) and (Y',Z') are two solutions to Equation (5.1) with $0 < Y, Y' \le Y^{h_1,h_2}$, then Y and Y' are indistinguishable processes, and $Z = Z'$ $d\mathbb{P} \times dt$-almost surely.

$Y^{\tilde h}$ *is the first component of the (maximal) solution to Equation (3.4) with parameters* $(X,\tilde h)$. *Specifically,* (Y,Z) *has the following regularity:*

$$\mathbb{E}\Big[\sup_{t\in[0,T]} Y_t^{p(2\delta+1)(e^B+1)}\Big] < +\infty \quad and \quad Z\in\mathcal{L}^2_T.$$

Moreover, if there exists $\varepsilon>0$ *such that* $X\geq\varepsilon$, *then* Z *has the further regularity* $Z\in\mathcal{M}^2_T$.
(ii) In addition, assume $\tilde f$ *can be decomposed as*

$$\tilde f(t,y,z)=y(\tilde f_1(t,y)+\tilde f_2(t,z)),$$

with $\tilde f_1,\tilde f_2>0$, $\tilde f_1(\cdot,y)$ *convex in* y, *and* $\tilde f_2(\cdot,z)$ *convex in* z. *Then, if* $\alpha,\beta\in\mathcal{H}^\infty_T$ *and* $p\geq\max\{\frac{2}{2\delta+1},1\}$, *Equation (3.4) admits a unique solution such that* $0<Y\leq Y^{\tilde h}$. *Therefore, when* $\tilde f$ *does not depend on* y *and is convex in* z, *then uniqueness holds without further assumptions.*

We finally provide a stability result for two-driver BSDEs in the spirit of Theorem 20.

Proposition 31. *With the same notation as in Proposition 28, consider* $(X^n,g_1^n,g_2^n)_{n\in\mathbb{N}}$, (X,g_1,g_2) *with* g_1, $\sup_{n\in\mathbb{N}}g_1^n$ *verifying A'),* g_2, $\sup_{n\in\mathbb{N}}g_2^n$ *verifying (G2) and* $(g_1^n,g_2^n)_{n\in\mathbb{N}}$, (g_1,g_2) *verifying C'). For any* $n\in\mathbb{N}$, *let* $X^n,X\geq\varepsilon$ *for some* $\varepsilon>0$ *and let* $X,\sup_{n\in\mathbb{N}}X^n\in L^{p(\frac{2\delta}{K^2}+1)(e^B+1)}$, *where* $p>\max\{\frac{2K^2}{2\delta+K^2},1\}$ *and* $B=(\|\alpha\|^T_\infty+\|\beta\|^T_\infty)T$. *We call* (Y,Z) *(resp.* $(Y^n,Z^n),n\in\mathbb{N}$*) the unique solution to the two-driver BSDE with parameters* (X,g_1,g_2) *(resp.* (X,g_1^n,g_2^n), $n\in\mathbb{N}$*) such that* $0\leq Y^n,Y\leq Y^{h_1,h_2}$. *Suppose there exists a constant* $C>0$ *such that for any* $n\in\mathbb{N}$ *the* z*-inverse* $(g_2^n)^{-1}$ *verifies* $d\mathbb{P}\times dt$*-a.s. and for any* $(y,z)\in\mathbb{R}_+\times\mathbb{R}^n$*:*

$$|(g_2^n)^{-1}(t,y_1,z_1)-(g_2^n)^{-1}(t,y_2,z_2)|\leq C\left|\frac{z_1}{y_1}-\frac{z_2}{y_2}\right|,$$

and $d\mathbb{P}\times dt$*-a.s.* $(g_2^n)^{-1}(t,y,z)\to g_2^{-1}(t,y,z)$ *for any* $(y,z)\in\mathbb{R}_+\times\mathbb{R}^n$.
Setting $V_\cdot:=g_2(\cdot,Y_\cdot,Z_\cdot)$, *if* $\int_0^T g_1^n(t,Y_t,(g_2^n)^{-1}(t,Y_t,V_t))dt\to\int_0^T g_1(t,Y_t,g_2^{-1}(t,Y_t,V_t))dt$ *as* $n\to\infty$ *in the sense of* $L^{p(\frac{2\delta}{K^2}+1)(e^B+1)}$*-convergence and* $X_n\to X$ $d\mathbb{P}$*-a.s., then* $(Y^n,Z^n)\xrightarrow{n\to\infty}(Y,Z)$ *in* $L^q\times\mathcal{M}^r_T$ *for any* $q\in[1,p(\frac{2\delta}{K^2}+1)(e^B+1))$ *and* $r\in[1,2p)$.

Proof. Akin to Proposition 26, we define $g^n(t,y,v):=g_1^n(t,y,(g_2^n)^{-1}(t,y,v))$ and $g(t,y,v):=g_1(t,y,g_2^{-1}(t,y,v))$. Thus, the ordinary BSDEs with parameters $(X^n,g^n)_{n\in\mathbb{N}}$ and (X,g) verify all the hypotheses of Theorem 20. Setting (Y,V) (resp. (Y^n,V^n)) the solution to the BSDE with parameters (X,g) (resp. (X^n,g^n)), Theorem 20 entails $(Y^n,V^n)\to(Y,V)$ in $L^q\times\mathcal{M}^{2p}_T$ for any $q\in[1,p(\frac{2\delta}{K^2}+1)(e^B+1))$. We need to check that $Z^n\to Z$ in $\mathcal{M}^r_T$ for any $r\in[1,2p)$. We know that $V_\cdot=g_2(\cdot,Y_\cdot,Z_\cdot)$ and $V^n_\cdot=g_2^n(\cdot,Y^n_\cdot,Z^n_\cdot)$, thus $Z_\cdot=g_2^{-1}(\cdot,Y_\cdot,V_\cdot)$ and $Z^n_\cdot=(g_2^n)^{-1}(\cdot,Y^n_\cdot,V^n_\cdot)$. By the triangular inequality, it holds that

$$|Z^n_t-Z_t|\leq|(g_2^n)^{-1}(t,Y^n_t,V^n_t)-(g_2^n)^{-1}(t,Y_t,V_t)|+|(g_2^n)^{-1}(t,Y_t,V_t)-g_2^{-1}(t,Y_t,V_t)|, \tag{5.6}$$

$d\mathbb{P}\times dt$-a.s. By the assumptions on $(g_2^n)^{-1}$, the first term in the right-hand member verifies

$$|(g_2^n)^{-1}(t,Y^n_t,V^n_t)-(g_2^n)^{-1}(t,Y_t,V_t)|\leq C\left|\frac{V^n_t}{Y^n_t}-\frac{V_t}{Y_t}\right|.$$

Since $Y^n,Y\geq\varepsilon$, $Y^n\to Y$ and $V^n\to V$ in probability, also $\frac{V^n}{Y^n}\to\frac{V}{Y}$ in probability, and the last equation leads to

$$(g_2^n)^{-1}(t,Y^n_t,V^n_t)\xrightarrow{n\to\infty}(g_2^n)^{-1}(t,Y_t,Z_t),$$

in probability. In addition, the second term in right-hand member of Equation (5.6) converges to 0, $d\mathbb{P}\times dt$-a.s., by the pointwise convergence of $(g_2^n)^{-1}$ to g_2^{-1}, hence also in probability. These convergences and Equation (5.6) entail that $Z^n \to Z$ in probability. Furthermore, the growth rate of g_2 yields

$$\sup_{n\in\mathbb{N}} \mathbb{E}\left[\left(\int_0^T |Z_t^n|^2 dt\right)^p\right] = \sup_{n\in\mathbb{N}} \mathbb{E}\left[\left(\int_0^T |(g_2^n)^{-1}(t, Y_t^n, V_t^n)|^2 dt\right)^p\right]$$
$$\leq K \sup_{n\in\mathbb{N}} \mathbb{E}\left[\left(\int_0^T \left|\frac{V_t^n}{Y_t^n}\right|^2 dt\right)^p\right] \leq K_\varepsilon \sup_{n\in\mathbb{N}} \mathbb{E}\left[\left(\int_0^T |V_t^n|^2 dt\right)^p\right] < +\infty,$$

where the last inequality follows from $Y^n \geq \varepsilon$ for any $n \in \mathbb{N}$, and the regularity of V^n (see Theorem 20). Hence, Z^n is uniformly integrable in $\mathcal{M}_T^r$ for any $r \in [1, 2p)$, and Vitali's convergence theorem gives the required convergence. □

Remark 32. *We underline that if $(X^n)_{n\in\mathbb{N}}, X$ are strictly positive but not necessarily bounded away from 0, then the convergence of $Y^n \to Y$ in $\mathcal{H}_T^q$ for any $q \in [1, p(2\delta+1)(e^B+1))$ is preserved, as is clear from the proof of Proposition 31, whereas the regularities for $(Z^n)_{n\in\mathbb{N}}$ and Z do not necessarily hold.*

6 Applications to Return and Star-Shaped Risk Measures

In this section, we apply the theoretical results on GBSDEs to study return and star-shaped risk measures. We begin by establishing existence and uniqueness, without requiring the restrictive condition $X \in \mathcal{L}^\infty(\mathcal{F}_T)$ of Section 3 on the terminal condition. Next, we systematically analyze the structural properties of the induced dynamic functionals. Each mathematical result is stated in a succinct, formal form, followed by a brief financial interpretation. Finally, three examples illustrate how specific star-shaped risk measures and (robust) L^p-norms are naturally embedded in our setting.

6.1 General setting and properties

We consider the GBSDE

$$\tilde{\rho}_t = X + \int_t^T \tilde{\rho}_s \tilde{f}(s, \tilde{\rho}_s, \tilde{Z}_s) ds - \int_t^T \tilde{\rho}_s \tilde{Z}_s dW_s, \tag{6.1}$$

where $X \in L_+^{p(2\delta+1)(e^B+1)}$ and $\tilde{f} : \Omega \times [0,T] \times \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ satisfies the growth condition

$$\tilde{f}(t, y, z) \leq \alpha_t + \beta_t |\ln(y)| + \delta |z|^2, \quad d\mathbb{P} \times dt\text{-a.s.}, \ \forall (y, z) \in \mathbb{R}_+ \times \mathbb{R}^n.$$

Proposition 33 (Existence and uniqueness)**.** *Let α, β, δ, p and $\tilde{f}$ satisfy the assumptions stated in the second part of Corollary 30. Then Equation (6.1) admits a unique positive solution $(\tilde{\rho}, \tilde{Z})$,*

$$\tilde{\rho} \in \mathcal{H}_T^{p(2\delta+1)(e^B+1)}, \qquad \tilde{Z} \in \mathcal{L}_T^2.$$

Proof. This follows immediately from Corollary 30. □

Existence and uniqueness guarantee that the dynamic evaluation $\tilde{\rho}$ is mathematically well posed. Equation (6.1) defines a dynamic evaluation on $L_+^{p(2\delta+1)(e^B+1)}$ without relying on the one-to-one correspondence of Sections 2.2 and 3.3, which generally fails for these spaces. Thus, the functional $\tilde{\rho}$ can be computed for a broad class of potentially unbounded payoffs, with its BSDE representation ensuring numerical tractability and practical applicability. Next, recall Definitions 2–3.

Proposition 34 (Monotonicity and Lebesgue property)**.** *Under the assumptions of Proposition 33, the family $(\tilde{\rho}_t)_{t\in[0,T]}$ satisfies the following properties:*

1. *For every $t \in [0,T]$, the mapping $\tilde{\rho}_t : L_+^{p(2\delta+1)(e^B+1)}(\mathcal{F}_T) \to L_+^{p(2\delta+1)(e^B+1)}(\mathcal{F}_t)$ is monotone.*

2. *For every $t \in [0,T]$, $\tilde{\rho}_t$ has the Lebesgue property: if $(X^n) \subset L_+^{p(2\delta+1)(e^B+1)}(\mathcal{F}_T)$, $\sup_n X^n \in L_+^{p(2\delta+1)(e^B+1)}(\mathcal{F}_T)$ and $X_n \to X$ $\mathbb{P}-$a.s., then $\tilde{\rho}_t(X^n) \to \tilde{\rho}_t(X)$ $\mathbb{P}$-a.s.*

Proof. Monotonicity of $(\tilde{\rho}_t)_{t\in[0,T]}$ w.r.t. its terminal condition follows from an application of Proposition 29. The Lebesgue property follows from Proposition 31 and Remark 32. □

Monotonicity aligns with first-order stochastic dominance: larger payoffs are never considered less attractive. The Lebesgue property ensures a form of robustness, as evaluations of approximate or truncated payoffs converge to the true value. This entails in particular that if a sequence of payoffs is acceptable, then its limit is also acceptable; see, e.g., [7, 30] for further insights.

In the following lemma, we introduce a simple denseness result that will be useful to prove several properties of dynamic risk measures.

Lemma 35 (Denseness of bounded elements)**.** *For every $m \geq 1$, $\mathcal{L}^\infty(\mathcal{F}_T)$ is dense in $L_+^m(\mathcal{F}_T)$. In particular, for $X \in L_+^m$ one may set $X^n := n \wedge X \vee \frac{1}{n}$ with $X^n \in \mathcal{L}^\infty$ and $X^n \to X$ in L^m.*

Proof. We claim that $\mathcal{L}^\infty(\mathcal{F}_T)$ is dense in $L_+^m(\mathcal{F}_T)$ for any $m \geq 1$. Indeed, let us fix $m \geq 1$ and $X \in L_+^m(\mathcal{F}_T)$. Defining for each $n \in \mathbb{N}$ the random variable $X^n := n \wedge X \vee \frac{1}{n}$, it is clear that $X^n \in \mathcal{L}^\infty(\mathcal{F}_T)$ for any $n \in \mathbb{N}$, $X^n \to X$ $d\mathbb{P}$-a.s., and $0 < X^n \leq X \vee 1 \in L_+^m(\mathcal{F}_T)$. By dominated convergence, it holds that $X^n \to X$ in $L^m(\mathcal{F}_T)$, and the denseness is proved. □

Proposition 36 (Multiplicative convexity (GA-convexity))**.** *Suppose $\tilde{f}$ satisfies the geometric-arithmetic (GA) convexity condition*

$$\tilde{f}(t, y_1^\lambda y_2^{1-\lambda}, \lambda z_1 + (1-\lambda)z_2) \leq \lambda \tilde{f}(t, y_1, z_1) + (1-\lambda)\tilde{f}(t, y_2, z_2).$$

Then for every $t \in [0,T]$ and all $X, Y \in L_+^{p(2\delta+1)(e^B+1)}$,

$$\tilde{\rho}_t(X^\lambda Y^{1-\lambda}) \leq \tilde{\rho}_t^\lambda(X)\tilde{\rho}_t^{1-\lambda}(Y), \qquad \forall \lambda \in [0,1].$$

Proof. We first fix $X \in \mathcal{L}^\infty(\mathcal{F}_T)$. Then, proceeding as in the proof of Proposition 5, the corresponding monetary risk measure, defined by the one-to-one correspondence for any $t \in [0,T]$ as $\rho_t(\bar{X}) := \exp(\tilde{\rho}_t(X))$ with $\bar{X} := \ln(X)$, has dynamics given by a BSDE with parameters $(\bar{X}, f)$, where $f(t,y,z) = \tilde{f}(t, e^y, z) + \frac{1}{2}|z|^2$ for any $(y,z) \in \mathbb{R} \times \mathbb{R}^n$. We prove convexity of f w.r.t. (y,z). Let us fix $\lambda \in [0,1]$ and $(y_1,z_1),(y_2,z_2) \in \mathbb{R} \times \mathbb{R}^n$. It holds that

$$\begin{aligned}
&f(t, \lambda y_1 + (1-\lambda)y_2, \lambda z_1 + (1-\lambda)z_2) \\
&= \tilde{f}(t, \exp(\lambda y_1 + (1-\lambda)y_2), \lambda z_1 + (1-\lambda)z_2) + \frac{1}{2}|\lambda z_1 + (1-\lambda)z_2|^2 \\
&\leq \lambda \tilde{f}(t, e^{y_1}, z_1) + (1-\lambda)\tilde{f}(t, e^{y_2}, z_2) + \frac{\lambda}{2}|z_1|^2 + \frac{1-\lambda}{2}|z_2|^2 \\
&= \lambda f(t, y_1, z_1) + (1-\lambda) f(t, y_2, z_2).
\end{aligned}$$

The usual comparison theorem for BSDEs yields convexity of ρ_t on $L^\infty(\mathcal{F}_T)$ for any $t \in [0,T]$. By Proposition A.1.1 (e), $\tilde{\rho}_t$ is multiplicatively convex on $\mathcal{L}^\infty(\mathcal{F}_T)$ for any $t \in [0,T]$.

We want to prove

$$\tilde{\rho}_t(X^\lambda Y^{1-\lambda}) \leq \tilde{\rho}_t^\lambda(X)\tilde{\rho}_t^{1-\lambda}(Y), \ \forall X, Y \in L_+^{p(2\delta+1)(e^B+1)}, \ \lambda \in [0,1], \ t \in [0,T],$$

for generic $X, Y \in L_+^{p(2\delta+1)(e^B+1)}$. Fix $\lambda \in [0,1]$, $X, Y \in L_+^{p(2\delta+1)(e^B+1)}$, and $(X^n)_{n\in\mathbb{N}}, (Y^n)_{n\in\mathbb{N}}$ such that $X^n \to X$ and $Y^n \to Y$ in $L_+^{p(2\delta+1)(e^B+1)}$. Such sequences exist as per Lemma 35. For each $n \in \mathbb{N}$, we call $\tilde{\rho}^n_\cdot(X^n)$ the solution to Equation (6.1) with parameters $(X^n, \tilde{f})$ and similarly for parameters $(Y^n, \tilde{f})$. Then, for each $n \in \mathbb{N}$, by the first part of the proof,

$$\tilde{\rho}_t^n((X^n)^\lambda (Y^n)^{1-\lambda}) \leq \tilde{\rho}_t^\lambda(X^n)\tilde{\rho}_t^{1-\lambda}(Y^n),\ t \in [0,T]. \tag{6.2}$$

We now want to let $n \to \infty$, proving that both members of the previous inequality converge a.s. (at least for subsequences), preserving the inequality and yielding the thesis. To see this, we aim to employ Proposition 31 (and Remark 32). Indeed, $(X^n)_{n\in\mathbb{N}}, (Y^n)_{n\in\mathbb{N}}$ verify all the assumptions stated in Proposition 31, while the driver $\tilde{f}$ does not depend on $n \in \mathbb{N}$. Thus, $\tilde{\rho}^n_\cdot(X^n) \to \tilde{\rho}_\cdot(X)$ and $\tilde{\rho}^n_\cdot(Y^n) \to \rho_\cdot(Y)$ in $\mathcal{H}_T^q$ for any $q \in [1, p(2\delta+1)(e^B+1))$. Hence, there exists a subsequence $(\tilde{\rho}^{n_k}_\cdot(X^{n_k}))_{k\in\mathbb{N}}$ (resp. $(\tilde{\rho}^{n_k}_\cdot(Y^{n_k}))_{k\in\mathbb{N}}$) converging to $\tilde{\rho}_\cdot(X)$ (resp. $\tilde{\rho}_\cdot(Y)$) $d\mathbb{P}$-a.s. for any $t \in [0,T]$. In addition, we note that $(X^n)^\lambda (Y^n)^{1-\lambda} \to X^\lambda Y^{1-\lambda}$ $d\mathbb{P}$-a.s. and $X^\lambda Y^{1-\lambda} \in L_+^{p(2\delta+1)(e^B+1)}$ by the interpolation inequality (i.e., a generalization of Hölder's inequality). Hence, extracting yet another subsequence (without renaming it), we have $\tilde{\rho}^{n_k}_\cdot((X^{n_k})^\lambda (Y^{n_k})^{1-\lambda}) \to \tilde{\rho}_\cdot(X^\lambda Y^{1-\lambda})$ $d\mathbb{P}$-a.s. for any $t \in [0,T]$. Taking the limit in Equation (6.2), the thesis follows. □

Remark 37. *It is a routine verification to show that, under the increasing monotonicity assumption in y, GA-convexity in (y,z) relaxes the stronger hypothesis of joint convexity in (y,z) for the driver $\tilde{f}$ (see also Example 40).*

Multiplicative convexity embodies diversification in terms of compound returns: mixtures of strategies are never penalized beyond the product of their separate evaluations. This is the natural convexity notion in multiplicative (return-based) frameworks. Multiplicative convexity has been firstly introduced in the context of risk measures by [9, 38] in the static setting. Here we extend the definition to the dynamic environment and show its relation with the driver of a GBSDE.

Proposition 38 (Positive homogeneity and star-shapedness)**.** *In the setting of Proposition 33:*

1. *If $\tilde{f}$ does not depend on y, then $\tilde{\rho}_t$ is positively homogeneous.*
2. *If $\tilde{f}$ is increasing in y, then $\tilde{\rho}_t$ is star-shaped.*

Proof. First, fix $X \in \mathcal{L}^\infty(\mathcal{F}_T)$ and $\eta_t \in \mathcal{L}^\infty(\mathcal{F}_t)$. If $\tilde{f}$ does not depend on y, also $f(t,z) = \tilde{f}(t,z) + \frac{1}{2}|z|^2$ does not depend on y, thus the family of monetary risk measures $(\rho_t)_{t\in[0,T]}$ corresponding to the first component of the solution to the BSDE with parameters $(\ln(X), f)$ is cash-additive (see, e.g., [7, 34]). Thus, the corresponding family of return risk measures $(\tilde{\rho}_t)_{t\in[0,T]}$ is positively homogeneous on $\mathcal{L}^\infty$ by Proposition A.1.1 (b). Using again the denseness argument, we conclude that positive homogeneity holds on $L_+^{p(2\delta+1)(e^B+1)}$.
Star-shapdeness can be proved exactly as multiplicative convexity, by recalling that if $\tilde{f}$ is increasing in y, then also $f(t,y,z) = \tilde{f}(t,e^y,z) + \frac{1}{2}|z|^2$ has the same monotonicity in y. Thus, the comparison theorem for BSDEs yields cash-superadditivity for $(\rho_t)_{t\in[0,T]}$, hence $(\tilde{\rho}_t)_{t\in[0,T]}$ is star-shaped by Proposition A.1.1 (c). □

Positive homogeneity reflects scale-invariance under leverage in frictionless settings. Star-shapedness, a weaker property, captures decreasing marginal impact of scaling, thus reflecting liquidity costs or nonlinear exposures for large positions. While positive homogeneity is a well-known property for risk measures (see e.g., [30]), star-shapedness has recently been introduced in [14, 38] (see also [39]) and extended to a more general, dynamic and BSDE setting in [40].

Proposition 39 (Time-consistency and normalization)**.** *In the setting of Proposition 33:*

1. The family $(\tilde{\rho}_t)_{t\in[0,T]}$ is time-consistent.

2. If $\tilde{f}(\cdot,1,0)\equiv 0$, then $\tilde{\rho}_t(1)=1$ for all $t\in[0,T]$.

Proof. As far as time-consistency is concerned, it is sufficient to prove the statement on $\mathcal{L}^\infty(\mathcal{F}_T)$ and then use the denseness argument. We note that the flow property of BSDEs (see, e.g., [26]) immediately implies time-consistency for the first component of the solution to the BSDE with parameters $(\ln(X), f)$, hence the return counterpart $(\tilde{\rho}_t)_{t\in[0,T]}$ is time-consistent on $\mathcal{L}^\infty(\mathcal{F}_T)$ by Proposition A.1.1 (g).
If $\tilde{f}(\cdot,1,0)\equiv 0$, then by direct inspection the couple $(\tilde{\rho},\tilde{Z})\equiv(1,0)$ is the unique solution to Equation (6.1) with terminal condition $X\equiv 1$, thus normalization follows. □

Time-consistency ensures a well-structured dynamic "pasting" property, whereby evaluations at intermediate times remain aligned with earlier decisions. This property has been thoroughly studied in the context of recursive utilities ([16, 25, 44]) and dynamic risk measures ([11]) and is naturally satisfied for BSDE-induced risk measures due to the flow property of BSDEs (see [7], among others, for further details). Normalization anchors the measure, ensuring the unit payoff is valued at one, consistent with its interpretation as numéraire.

6.2 Examples of dynamic return and star-shaped risk measures

In the following example, we consider a natural, dynamic star-shaped risk measure; the two subsequent examples analyze canonical dynamic return risk measures.

Example 40. *Let us consider $\tilde{f}(t,y)=\beta_t\ln(1+y)$, where β is a positive and bounded stochastic process. Clearly, $\tilde{f}$ is increasing in y, strictly positive, the mapping $y\mapsto y\tilde{f}(t,y)$ is $d\mathbb{P}\times dt$-a.s. convex, and the respective growth condition is satisfied. Moreover, one can directly verify that $\tilde{f}$ is GA-convex (despite being concave).*

Now take a terminal condition $X\in L_+^{p(e^B+1)}$ for some $p>2$ and $B=2T\|\beta\|_\infty^T$. The GBSDE

$$\tilde{\rho}_t = X+\int_t^T \beta_s\tilde{\rho}_s\ln(1+\tilde{\rho}_s)\,ds-\int_t^T \tilde{\rho}_s\tilde{Z}_s\,dW_s,$$

admits a unique solution, as per Proposition 33. Hence, the family of functionals $(\tilde{\rho}_t)_{t\in[0,T]}$,

$$\tilde{\rho}_t : L_+^{p(e^B+1)}(\mathcal{F}_T)\to L_+^{p(e^B+1)}(\mathcal{F}_t),$$

is monotone, star-shaped, time-consistent, and multiplicatively convex, in accordance with Propositions 34, 36, 38, and 39. (The theory developed also applies in the special case $\delta\equiv 0$, as here.)

This prototypical GBSDE may be particularly suitable for modeling risk/pricing in the presence of ambiguous interest rates. Indeed, by analogy to the monetary case, ambiguity on interest rates translates into a lack of cash additivity (see, e.g., [27, 39]), which, in the multiplicative setting, corresponds to a lack of positive homogeneity. In this framework, star-shapedness replaces positive homogeneity, and β can be interpreted as a stochastic discount rate.

We now show that any dynamic (robust) γ-norm, with $\gamma>1$, admits a unique representation as a solution to a GBSDE. Thus, applying our theoretical results, we can describe the dynamics of general (robust) norms in L^p-spaces in terms of GBSDEs. Hence, while the dynamics of entropic risk measures are given by ordinary BSDEs (e.g., [7, 43, 45]), their return counterparts ([38]) follow GBSDEs. For ease of exposition, we start with the simpler case of dynamic γ-norms.

Example 41 (γ-norms)**.** *Consider $\gamma>1$ and $X\in L_+^{p\gamma}$ for some $p>1$. The following GBSDE admits at least one positive solution $(Y,Z)\in\mathcal{H}_T^{p\gamma}\times\mathcal{L}_T^2$ by Corollary 30 (i):*

$$Y_t = X+\int_t^T \frac{\gamma-1}{2}Y_s|Z_s|^2ds-\int_t^T Y_sZ_sdW_s. \tag{6.3}$$

We claim that the explicit formula for Y is given by $Y_t = \mathbb{E}\left[X^\gamma \middle| \mathcal{F}_t\right]^{\frac{1}{\gamma}}$, $d\mathbb{P} \times dt$-a.s. To see this, apply Itô's formula to $f(Y_\cdot) := Y_\cdot^\gamma$. It holds that

$$Y_t^\gamma = X^\gamma - \int_t^T V_s dW_s, \tag{6.4}$$

where $V := \gamma Y^\gamma Z$. Taking the conditional expectation and raising to power $1/\gamma$ both members of the previous equation, we obtain the thesis. In addition, uniqueness of the solution follows from uniqueness of the solution to Equation (6.4). In other words, we are able to fully characterize the dynamics of γ-norms as the unique solution to Equation (6.3) for any $\gamma > 1$. We note that in this simple case, monotonicity, positive homogeneity, multiplicative convexity and time-consistency can be verified by direct inspection.

Example 42 (Robust γ-norms)**.** *Consider $\gamma > 1$ and $X \in L_+^{p\gamma}$ for some $p \geq 2$. The following GBSDE admits at least one positive solution $(Y, Z) \in \mathcal{H}_T^{p\gamma} \times \mathcal{L}_T^2$, as per Corollary 30 (i):*

$$Y_t = X + \int_t^T Y_s \left(g(s, Z_s) + \frac{\gamma - 1}{2} |Z_s|^2 \right) ds - \int_t^T Y_s Z_s dW_s, \tag{6.5}$$

where $g : [0, T] \times \Omega \times \mathbb{R}^n \to \mathbb{R}_+$ is a $\mathcal{P} \times \mathcal{B}(\mathbb{R}^n)/\mathcal{B}(\mathbb{R}_+)$-measurable, convex and positively homogeneous function such that there exists $C > 0$ verifying $d\mathbb{P} \times dt$-a.s. $0 \leq g(t, z) \leq C|z|$, for any $z \in \mathbb{R}^n$. In fact, the explicit formula for Y is given by

$$Y_t = \sup_{\mu \in \mathcal{A}} \mathbb{E}_{\mathbb{Q}^\mu}\left[X^\gamma \middle| \mathcal{F}_t\right]^{\frac{1}{\gamma}}, \ d\mathbb{P} \times dt\text{-a.s.} \tag{6.6}$$

Here, $\mathcal{A} := \left\{ (\mu_t)_{t \in [0,T]} : |\mu_t| \leq C \ d\mathbb{P} \times dt\text{-a.s.} \right\}$, and $\mathbb{Q}^\mu$ is the probability measure with density $\mathcal{E}_T^\mu$. To see this, we apply Itô's formula to $f(Y_\cdot) := Y_\cdot^\gamma$. It holds that:

$$Y_t^\gamma = X^\gamma + \int_t^T g(s, V_s) ds - \int_t^T V_s dW_s.$$

Here, we used positive homogeneity of g, setting $V := \gamma Y^\gamma Z$. Since g does not depend on y and is sublinear in z, this BSDE admits a unique solution $(Y^\gamma, V) \in \mathcal{H}_T^p \times \mathcal{M}_T^p$, whose first component can be represented as (see, e.g., [7, 24]):

$$Y_t^\gamma = \sup_{\mu \in \mathcal{A}} \mathbb{E}_{\mathbb{Q}^\mu}\left[X^\gamma \middle| \mathcal{F}_t\right], \ d\mathbb{P} \times dt\text{-a.s.}$$

Raising to power $1/\gamma$ both members of the previous equation we verify Equation (6.6) and thus fully characterize the dynamics of robust γ-norms as the unique solution to Equation (6.5) for any $\gamma > 1$. The (dynamic) robust γ-norms are the natural return counterparts of (dynamic) entropy coherent risk measures, studied in [42] in a static setting. Let us finally remark that Y_t is a dynamic, normalized, multiplicatively convex and time-consistent return risk measure for any $t \in [0, T]$, in agreement with Propositions 34, 36, 38, and 39.

Acknowledgements. We are very grateful to Fabio Bellini, Sonja Cox, Freddy Delbaen, Marco Frittelli, Michael Kupper, Antonis Papapantoleon, Mitja Stadje and to conference participants at the 2024 Bachelier Finance Society World Congress in Rio de Janeiro, the 2024 Probability Conference in Rome and the 2025 General AMaMeF Conference in Verona for useful comments and discussions. This research was funded in part by the Netherlands Organization for Scientific Research under grant NWO Vici 2020–2027 (Laeven) and by an Ermenegildo Zegna Founder's Scholarship (Zullino). Emanuela Rosazza Gianin and Marco Zullino are members of Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA, INdAM), Italy, and acknowledge the financial support of Gnampa Research Project 2024 (PRR-20231026-073916-203).

References

[1] Aumann, R. J. and R. Serrano (2008). An economic index of riskiness. *Journal of Political Economy* 116(5), 810-836.

[2] Bahlali, K. (2020). A domination method for solving unbounded quadratic BSDEs. *Graduate Journal of Mathematics* 5, 20-36.

[3] Bahlali, K., M. Eddahbi and Y. Ouknine (2017). Quadratic BSDE with $\mathbb{L}^2$-terminal data: Krylov's estimate, Itô-Krylov's formula and existence results. *The Annals of Probability* 45(4), 2377-2397.

[4] Bahlali, K., E. Essaky and M. Hassani (2015). Existence and uniqueness of multidimensional BSDEs and of systems of degenerate PDEs with superlinear growth generator. *SIAM Journal on Mathematical Analysis* 47(6), 4251-4288.

[5] Bahlali, K., O. Kebiri, N. Khelfallah and H. Moussaoui (2017). One dimensional BSDEs with logarithmic growth application to PDEs. *Stochastics* 89, 1061-1081.

[6] Bahlali, K. and L. Tangpi (2021). BSDEs driven by $|z|^2/y$ and applications to PDEs and decision theory. *Preprint on arXiv: https://arxiv.org/abs/1810.05664v3.*

[7] Barrieu, P. and N. El Karoui (2005). Pricing, hedging and optimally designing derivatives via minimization of risk measures. In: Carmona, R. (ed.). *Indifference Pricing: Theory and Applications*, 77-144, Princeton University Press, Princeton.

[8] Bauer, D. and G. Zanjani (2016). The marginal cost of risk, risk measures, and capital allocation. *Management Science* 62(5), 1431-1457.

[9] Bellini, F., R. J. A. Laeven and E. Rosazza Gianin (2018). Robust return risk measures. *Mathematics and Financial Economics* 12, 5-32.

[10] Bellini F., R. J. A. Laeven and E. Rosazza Gianin (2021). Dynamic robust Orlicz premia and Haezendonk-Goovaerts risk measures. *European Journal of Operational Research* 291, 438-446.

[11] Bion-Nadal, J. (2008). Dynamic risk measures: Time consistency and risk measures from BMO martingales. *Finance and Stochastics* 12, 219-244.

[12] Briand, P. and Y. Hu (2006). BSDE with quadratic growth and unbounded terminal value. *Probability Theory and Related Fields* 136, 604-618.

[13] Briand, P. and Y. Hu (2008). Quadratic BSDEs with convex generators and unbounded terminal conditions. *Probability Theory and Related Fields* 141, 543-567.

[14] Castagnoli, E., G. Cattelan, F. Maccheroni, C. Tebaldi and R. Wang (2022). Star-shaped risk measures. *Operations Research* 70(5), 2637-2654.

[15] Chateauneuf, A. and J. H. Faro (2010). Ambiguity through confidence functions. *Journal of Mathematical Economics* 45, 535-558.

[16] Chen, Z. and L. G. Epstein (2002). Ambiguity, risk, and asset returns in continuous time. *Econometrica* 70, 1403-1443.

[17] Coquet, F., Y. Hu, J. Mémin and S. Peng (2002). Filtration-consistent nonlinear expectations and related g-expectations. *Probability Theory and Related Fields* 123(1), 1-27.

[18] Delbaen, F. (2012). *Monetary Utility Functions.* Osaka University Press, Osaka.

[19] Delbaen, F., S. Peng and E. Rosazza Gianin (2010). Representation of the penalty term of dynamic concave utilities. *Finance and Stochastics* 14, 449-472.

[20] Delbaen, F., Y. Hu and A. Richou (2011). On the uniqueness of solutions to quadratic BSDEs with convex generators and unbounded terminal conditions. *Annales de l'Institut Henri Poincaré – Probabilités et Statistiques* 47(2), 559-574.

[21] Detemple, J. and M. Rindisbacher (2009). Dynamic asset allocation: Portfolio decomposition formula and applications. *The Review of Financial Studies* 23(1), 25-100.

[22] Drapeau, S., G. Heyne and M. Kupper (2013). Minimal supersolutions of convex BSDEs. *The Annals of Probability* 41, 3973-4001.

[23] Drapeau, S. and M. Kupper (2013). Risk preferences and their robust representation. *Mathematics of Operations Research* 38, 28-62.

[24] Drapeau, S., M. Kupper, E. Rosazza Gianin and L. Tangpi (2016). Dual representation of minimal supersolutions of convex BSDEs. *Annales de l'Institut Henri Poincaré – Probabilités et Statistiques* 52, 868-887.

[25] Duffie, D. and L. G. Epstein (1992). Stochastic differential utility. *Econometrica* 60, 353-394.

[26] El Karoui, N., S. Peng and M. C. Quenez (1997). Backward stochastic differential equations in finance. *Mathematical Finance* 7, 1-71.

[27] El Karoui, N. and C. Ravanelli (2009). Cash subadditive risk measures and interest rate ambiguity. *Mathematical Finance* 19, 561-590.

[28] Ethier, S. N. and T. G. Kurtz (1986). *Markov Process: Characterization and Convergence*. J. Wiley, New York.

[29] Fan, S., Y. Hu and S. Tang (2023). Existence, uniqueness and comparison theorem on unbounded solutions of scalar super-linear BSDEs. *Stochastic Processes and Their Applications* 157, 335-375.

[30] Föllmer, H. and A. Schied (2011). *Stochastic Finance*. 3rd ed., De Gruyter, Berlin.

[31] Foster, D. P. and S. Hart (2009). An operational measure of riskiness. *Journal of Political Economy* 117(5), 785-814.

[32] Hu, Y., P. Imkeller and M. Müller (2005). Utility maximization in incomplete markets. *The Annals of Applied Probability* 15(3), 1691-1712.

[33] Hun, O., M.-C. Kim and K.-G. Kim (2022). Stochastic Bihari inequality and applications to BSDE. *Journal of Mathematical Analysis and Applications* 513(1), 126204.

[34] Jiang, L. (2008). Convexity, translation invariance and subadditivity for g-expectations and related risk measures. *The Annals of Applied Probability* 18(1), 245-258.

[35] Kazamaki, N. (1994). *Continuous Exponential Martingales and BMO*. Lecture Notes in Mathematics 1579, Springer, Berlin.

[36] Kobylanski, M. (2000). Backward stochastic differential equations and partial differential equations with quadratic growth. *The Annals of Probability* 28(2), 558-602.

[37] Krätschmer, V., M. Ladkau, R. J. A. Laeven, J. G. M. Schoenmakers and M. A. Stadje (2018). Optimal stopping under uncertainty in drift and jump intensity. *Mathematics of Operations Research* 43, 1177-1209.

[38] Laeven, R. J. A. and E. Rosazza Gianin (2022). Quasi-logconvex measures of risk. *Preprint on arXiv:2208.07694v1.*

[39] Laeven, R. J. A., E. Rosazza Gianin and M. Zullino (2024). Law-invariant return and star-shaped risk measures. *Insurance: Mathematics and Economics* 117, 140-153.

[40] Laeven, R. J. A., E. Rosazza Gianin and M. Zullino (2025). Dynamic return and star-shaped risk measures via BSDEs. *Finance and Stochastics*, in press.

[41] Laeven, R. J. A., J. G. M. Schoenmakers, N. F. F. Schweizer and M. A. Stadje (2025). Robust multiple stopping — A duality approach. *Mathematics of Operations Research*, 50, 1250-1276.

[42] Laeven, R. J. A. and M. A. Stadje (2013). Entropy coherent and entropy convex measures of risk. *Mathematics of Operations Research* 38, 265-293.

[43] Laeven, R. J. A. and M. A. Stadje (2014). Robust portfolio choice and indifference valuation. *Mathematics of Operations Research* 39, 1109-1141.

[44] Lazrak, A. and M. C. Quenez (2003). A generalized stochastic differential utility. *Mathematics of Operations Research* 28, 154-180.

[45] Mania, M. and M. Schweizer (2005). Dynamic exponential utility indifference valuation. *The Annals of Applied Probability* 15(3), 2113-2143.

[46] Papapantoleon, A., D. Possamaï and A. Saplaouras (2023). Stability of backward stochastic differential equations: The general Lipschitz case. *Electronic Journal of Probability* 28, 1–56.

[47] Pardoux, E. and S. Peng (1990). Adapted solution of a backward stochastic differential equation. *Systems and Control Letters* 14, 55-61.

[48] Peng, S. (1997). Backward SDE and related g-expectation. In: El Karoui, N. and L. Mazliak (eds.). *Backward Stochastic Differential Equations*, 141-159, Longman, Harlow.

[49] Peng, S. (2005). Dynamically consistent nonlinear evaluations and expectations. *Preprint on arXiv:math.PR/0501415.*

[50] Pratt, J. W. (1964). Risk aversion in the small and in the large. *Econometrica* 32, 122-136.

[51] Protter, P. (2005). *Stochastic Integration and Differential Equations.* Springer Science & Business Media.

[52] Rosazza Gianin, E. (2006). Risk measures via g-expectations. *Insurance: Mathematics and Economics* 39, 19-34.

[53] Wakker, P. P. (2008). Explaining the characteristics of the power (CRRA) utility family. *Health Economics* 17, 1329-1344.

[54] Wang, X. and S. Fan (2018). A class of stochastic Gronwall's inequality and its application. *Journal of Inequalities and Applications* 336, 1-10.

[55] Zhang, J. (2017). *Backward Stochastic Differential Equations.* Springer, New York, NY.

SUPPLEMENTARY MATERIAL TO “Geometric BSDEs” (FOR ONLINE PUBLICATION)

A Online Appendix

A.1 Additional material to Section 2.2

Proposition A.1.1. *For any $t \in [0,T]$, let $\tilde{\rho}_t$ and ρ_t be as defined in (2.1) and (2.2). Then:*
(a) $\rho_t(0) = 0$ (normalization for ρ) if and only if $\tilde{\rho}_t(1) = 1$ (normalization for $\tilde{\rho}$) for any $t \in [0,T]$.
(b) ρ_t satisfies cash-additivity on $L^\infty(\mathcal{F}_T)$ if and only if $\tilde{\rho}_t$ satisfies positive homogeneity on $\mathcal{L}^\infty(\mathcal{F}_T)$.
(c) ρ_t satisfies cash-super-additivity on $L^\infty(\mathcal{F}_T)$, i.e., $\rho_t(X+m_t) \geq \rho_t(X) + m_t$ for any $X \in L^\infty(\mathcal{F}_T)$ and $m_t \in L^\infty_+(\mathcal{F}_t)$, if and only if $\tilde{\rho}_t$ satisfies star-shapedness on $\mathcal{L}^\infty(\mathcal{F}_T)$.
(d) ρ_t is monotone on $L^\infty(\mathcal{F}_T)$ if and only if $\tilde{\rho}_t$ is monotone on $\mathcal{L}^\infty(\mathcal{F}_T)$.
(e) ρ_t is convex on $L^\infty(\mathcal{F}_T)$ if and only if $\tilde{\rho}_t$ is multiplicatively convex on $\mathcal{L}^\infty(\mathcal{F}_T)$.
(f) ρ_t is positively homogeneous on $L^\infty(\mathcal{F}_T)$ if and only if $\tilde{\rho}_t$ is multiplicatively positively homogeneous on $\mathcal{L}^\infty(\mathcal{F}_T)$, i.e., for any $t \in [0,T]$, $X \in \mathcal{L}^\infty(\mathcal{F}_T)$ and $\eta_t \in \mathcal{L}^\infty(\mathcal{F}_t)$, $\tilde{\rho}_t(X^{\eta_t}) = \tilde{\rho}_t^{\eta_t}(X)$.
(g) The family of functionals $(\rho_t)_{t\in[0,T]}$ is time-consistent on $L^\infty(\mathcal{F}_T)$ if and only if $(\tilde{\rho}_t)_{t\in[0,T]}$ is time-consistent on $\mathcal{L}^\infty(\mathcal{F}_T)$.

Proof. Items (a)-(g) can easily be obtained by applying (2.1) and (2.2). □

A.2 Proofs

Proof of Proposition 5. For each fixed $X \in \mathcal{L}^\infty(\mathcal{F}_T)$, set $\rho_t(\ln(X)) := \ln \tilde{\rho}_t(X)$, whose dynamics are given by Itô's formula:

$$\rho_t = \ln(X) + \int_t^T \left(\tilde{f}((s, e^{\rho_s}, \tilde{Z}_s) + \frac{1}{2}|\tilde{Z}_s|^2 \right) ds - \int_t^T \tilde{Z}_s dW_s.$$

We define $f : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ by the formula

$$f(t,y,z) := \tilde{f}(t, e^y, z) + \frac{1}{2}|z|^2.$$

It is easy to check that A1) and A2) are verified. Indeed, R2) yields

$$|f(t,y,z)| = \left| \tilde{f}(t, e^y, z) + \frac{1}{2}|z|^2 \right| \leq C(1 + |\ln(e^y)|) + \left(C + \frac{1}{2} \right)|z|^2 \leq C'(1 + |y| + |z|^2).$$

Clearly, all measurability properties of $\tilde{f}$ stated in R1) are inherited by f. Thus, f verifies A1) and A2) and we conclude that there exists a solution (Y,Z) with the required regularity. In addition, when R3) is assumed on $\tilde{f}$, it easy to check that f verifies A3). Thus, uniqueness follows. Monotonicity w.r.t. X follows from monotonicity of logarithmic and exponential functions and the comparison principle for quadratic BSDEs (see Theorem 7.3.1 of [55]). □

Proof of Proposition 8. We start by proving the first statement. Fix $X \in L^0_+(\mathcal{F}_T)$ and define $f(x) := \ln(x)$. Applying Itô's formula to $Y'_\cdot = f(Y_\cdot)$ we obtain

$$-dY'_t = e^{-Y'_t} \left(g(t, e^{Y'_t}, Z_t) + \frac{e^{-Y'_t}|Z_t|^2}{2} \right) - e^{-Y'_t} Z_t dW_t.$$

Setting $Z'_t := e^{-Y'_t} Z_t$ it holds that

$$-dY'_t = e^{-Y'_t} \left(g(t, e^{Y'_t}, e^{Y'_t} Z'_t) + \frac{e^{Y'_t}|Z'_t|^2}{2} \right) - Z'_t dW_t,$$

with terminal condition $X' = \ln(X)$. By defining $g'(t,y,z) := e^{-y}g(t,e^y,e^y z) + \frac{|z|^2}{2}$, it results that

$$|g'(t,y,z)| \le e^{-y}\left(\alpha_t e^y + \beta_t e^y |\ln(e^y)| + \delta \frac{e^{2y}|z|^2}{e^y}\right) + \frac{|z|^2}{2} = \alpha_t + \beta_t|y| + \left(\delta + \frac{1}{2}\right)|z|^2,$$

thus the BSDE with parameters (X', g') admits a solution $(Y', Z') := (\ln(Y), Z/Y)$, in the sense of Definition 4. Indeed, Y is a positive and continuous process, as composition of continuous functions in their domain, $\int_0^T |Z'_s|^2 ds \le \frac{1}{\operatorname{ess\,inf}_t Y_t}\int_0^T |Z_s|^2 ds < +\infty$ and

$$\int_0^T |g'(s, Y'_s, Z'_s)|ds \le \int_0^T \alpha_s + \beta_s|Y'_s| + \left(\delta + \frac{1}{2}\right)|Z'_s|^2 ds < +\infty.$$

The other implication can be proved analogously by employing the substitution $Y := \exp(Y')$, verifying that $(Y, Z) = (\exp(Y'), \exp(Y')Z')$ is a solution to the BSDE (3.5) with parameters (X, g), where $X := \exp(X')$. □

Proof of Corollary 9. Since $X \in \mathcal{L}^\infty(\mathcal{F}_T)$, $X' := \ln(X) \in L^\infty(\mathcal{F}_T)$, hence the corresponding quadratic BSDE with parameters (X', g') with g' as in Proposition 8 admits a maximal and minimal solution verifying $(Y', Z') \in \mathcal{H}_T^\infty \times \mathrm{BMO}(\mathbb{P})$, by results in [36]. Thus, $(Y, Z) := (\exp(Y'), \exp(Y')Z') \in \mathcal{H}_T^\infty \times \mathrm{BMO}(\mathbb{P})$ is a solution to the BSDE with parameters (X, g). □

Proof of Proposition 10. (i) Consider

$$-d\bar{Y}_t = \left((2\delta+1)\alpha_t \bar{Y}_t + \beta_t \bar{Y}_t |\ln(\bar{Y}_t)|\right) dt - \bar{Z}_t dW_t, \tag{A.I}$$

with terminal condition $\bar{X} = X^{2\delta+1}$. We define

$$\bar{h}(t,y,z) := (2\delta+1)\alpha_t y + \beta_t y |\ln(y)|, \quad d\mathbb{P} \times dt\text{-a.s.}, \forall (y,z) \in \mathbb{R}_+ \times \mathbb{R}^n.$$

Equation (A.I) admits a positive solution if and only if Equation (3.5) does. To see this, it is enough to apply Itô's formula to $\bar{Y} = u(Y)$ to get Equation (A.I) from Equation (3.5) and *vice versa* for $Y = u^{-1}(\bar{Y})$, where $u(x) := x^{2\delta+1}$ for any $x > 0$.

The existence of a positive solution to Equation (A.I) is equivalent to proving the existence of a positive solution (w, ζ) to

$$w_t = \ln(1+\bar{X}) + \int_t^T \tilde{h}(s, w_s, \zeta_s)ds - \int_t^T \zeta_s dW_s, \tag{A.II}$$

where, $d\mathbb{P} \times dt$-a.s., $\forall (y,z) \in \mathbb{R}_+ \times \mathbb{R}^n$,

$$\tilde{h}(t,y,z) := ((2\delta+1)\alpha_s(e^y - 1) + \beta_s(e^y-1)|\ln(e^y-1)|)e^{-y} + \frac{1}{2}|z|^2.$$

To see this, it suffices to apply Itô's formula to $w := \ln(1+\bar{Y})$ and $\bar{Y} = \exp(w) - 1$. Since $\frac{x|\ln(x)|}{x+1} \le 1 + |\ln(x+1)|$, it holds that

$$0 \le \tilde{h}(t,y,z) \le (2\delta+1)\alpha_t + \beta_t + \beta_t y + \frac{1}{2}|z|^2, \quad d\mathbb{P} \times dt\text{-a.s.}, \forall (y,z) \in \mathbb{R}_+ \times \mathbb{R}^n.$$

As $\tilde{X} := \ln(1+\bar{X}) > 0$, by Theorem 3.1 (i) of [2], there exists a positive solution to the Equation (A.II) with parameters $(\tilde{X}, \tilde{h})$, yielding the thesis. In addition, the regularity of w provided by Theorem 3.1 (i) yields the required regularity for $Y = (\exp(w) - 1)^{\frac{1}{2\delta+1}}$.

(ii) Now we assume (H1)+(H2)'. Clearly, existence of the solution follows as above, while the regularity of Y is a consequence of Proposition 3.2 in [2]. We prove that $Z \in \mathcal{M}_T^2$. We can use a similar technique as in Proposition 3.5 of [6].

Let us start by considering the case $\delta \neq \frac{1}{2}$. We apply Itô's formula to $f(x) = x^2$, obtaining

$$Y_t^2 = X^2 + \int_t^T \left(2\alpha_s Y_s^2 + 2\beta_s Y_s^2 |\ln(Y_s)| + (2\delta - 1)|Z_s^2|\right) ds - 2\int_t^T Y_s Z_s dW_s.$$

Considering the sequence of stopping times $\tau_n := \inf\{t > 0 : \int_0^t 4|Y_s|^2 |Z_s|^2 ds \geq n\}$, rearranging and taking the expectation, it holds that

$$\begin{aligned}|2\delta - 1|\mathbb{E}\left[\int_0^{T\wedge\tau_n} |Z_t|^2 dt\right] &\leq \mathbb{E}\left[X^2 + Y_0^2 + \int_0^{T\wedge\tau_n} \left(2\alpha_t Y_t^2 + 2\beta_t Y_t^2 |\ln(Y_t)|\right) dt\right] \\ &\leq \mathbb{E}\left[X^2 + Y_0^2 + \int_0^{T\wedge\tau_n} \left(\frac{2\alpha_t^q}{q} + \frac{2Y_t^{2p}}{p} + \frac{2\beta_t^q}{q} + \frac{2Y_t^{2p}|\ln(Y_t)|^p}{p}\right) dt\right],\end{aligned}$$

where the last inequality follows from Young's inequality. In order to study the integrability of Z, we make use of the inequality $y^{2p}|\ln(y)|^p \leq K_\varepsilon + y^{2p+\varepsilon}$ that holds for any $\varepsilon > 0$, with $K_\varepsilon > 0$ depending only on the chosen $\varepsilon > 0$. This yields

$$|2\delta - 1|\mathbb{E}\left[\int_0^{T\wedge\tau_n} |Z_t|^2 dt\right] \leq K_\varepsilon \mathbb{E}\left[\int_0^T \left(\frac{2\alpha_t^q}{q} + \frac{2\beta_t^q}{q}\right) dt + K_T \sup_{t\in[0,T]} \left\{Y_t^{2p+\varepsilon}\right\}\right],$$

where $K_T > 0$. Letting $n \to \infty$, we have $T \wedge \tau_n \to T$ $d\mathbb{P}$-a.s., and Fatou's lemma yields

$$|2\delta - 1|\mathbb{E}\left[\int_0^T |Z_t|^2 dt\right] \leq K_\varepsilon \mathbb{E}\left[\int_0^T \left(\frac{2\alpha_t^q}{q} + \frac{2\beta_t^q}{q}\right) dt + K_T \sup_{t\in[0,T]} \left\{Y_t^{2p+\varepsilon}\right\}\right].$$

Choosing $\varepsilon = 2\delta p > 0$, we obtain the thesis, since $2p + \varepsilon = 2p(\delta + 1) \leq p(2\delta + 1)(e^{\inf_{\omega\in\Omega} B} + 1)$.

If $\delta = \frac{1}{2}$, Itô's formula applied to the function $x^2 \ln(x)$ entails

$$\begin{aligned}Y_t^2 \ln(Y_t) = X^2 \ln(X) \\ &+ \int_t^T \left(\alpha_s Y_s^2(1 + 2\ln(Y_s)) + \beta_s Y_s^2 |\ln(Y_s)|(1 + 2\ln(Y_s)) - |Z_t|^2\right) ds \\ &- \int_t^T (2Y_s \ln(Y_s) + Y_s) Z_s dW_s.\end{aligned}$$

Considering the sequence of stopping times $\tau_n := \inf\{t > 0 : \int_0^t |Y_s Z_s|^2 + |Y_s \ln(Y_s) Z_s|^2 ds \geq n\}$ and arguing similarly as above we get

$$\mathbb{E}\left[\int_0^T |Z_t|^2 dt\right] \leq K\mathbb{E}\left[\int_0^T \left(\frac{2\alpha_t^q}{q} + \frac{2\beta_t^q}{q}\right) dt + \sup_{t\in[0,T]} \left\{Y_t^{2p(\delta+1)}\right\}\right] < +\infty,$$

for some $K > 0$.

(iii) Finally, consider (H1)'+(H2)". Following the same passages as above and including the new term $\gamma_t|z|$ in the driver, we obtain the transformed equation

$$w_t' = \ln(1 + \bar{X}) + \int_t^T h'(s, w_s', \zeta_s') ds - \int_t^T \zeta_s' dW_s, \tag{A.III}$$

where

$$\begin{aligned}h'(t, y, z) &:= ((2\delta + 1)\alpha_s(e^y - 1) + \beta_s(e^y - 1)|\ln(e^y - 1)|)e^{-y} + \gamma_t|z| + \frac{1}{2}|z|^2 \\ &\leq (2\delta + 1)\alpha_t + \beta_t + \beta_t y + \gamma_t|z| + \frac{1}{2}|z|^2.\end{aligned}$$

By the proof of Proposition 3.3 in [2], Equation (A.III) admits a positive solution if the BSDEs with parameters $(\xi, h'') = (\exp((\bar{X} + \int_0^T \alpha_s + \beta_s ds)(e^{\int_0^T \beta_s ds} + 1)) - 1, \gamma.|z|)$ admits a positive solution (Y', Z'). By assumption (H2)" and Theorem 2.1 in [4], the BSDE with parameters (ξ, h'') admits a (unique) solution $(Y', Z') \in \mathcal{H}_T^p \times \mathcal{M}_T^p$. In addition, $Y' > 0$ since $\xi > 0$. Indeed, after choosing a suitable localization $(\tau_n')_{n\in\mathbb{N}}$ as above, it holds that

$$Y_t' = \mathbb{E}\left[\xi + \int_t^{T\wedge\tau_n'} \gamma_s |Z_s'| ds \,\middle|\, \mathcal{F}_t \right] \geq \mathbb{E}[\xi|\mathcal{F}_t] > 0,$$

where we employed the positivity of $\gamma|Z'|$. Thus, the thesis follows with the required regularity for the y-component of the solution.

It remains to be shown that $Z \in \mathcal{M}_T^2$. Suppose $\delta \neq 1/2$ and apply Itô's formula to $f(x) = x^2$, which yields

$$\begin{aligned} Y_t^2 = X^2 &+ \int_t^T \left(2\alpha_s Y_s^2 + 2\beta_s Y_s^2 |\ln(Y_s)| + 2\gamma_s Y_s |Z_s| + (2\delta - 1)|Z_s^2|\right) ds \\ &- 2\int_t^T Y_s Z_s dW_s. \end{aligned}$$

Taking τ_n as above and performing similar calculations, it holds that

$$\begin{aligned} &|2\delta - 1| \mathbb{E}\left[\int_0^{T\wedge\tau_n} |Z_t|^2 dt\right] \\ &\leq \mathbb{E}\left[X^2 + Y_0^2 + \int_0^{T\wedge\tau_n} \left(2\alpha_t Y_t^2 + 2\gamma_t Y_t |Z_t| + 2\beta_t Y_t^2 |\ln(Y_t)|\right) dt\right] \\ &\leq \mathbb{E}\Bigg[X^2 + Y_0^2 \\ &\quad + \int_0^{T\wedge\tau_n} \left(\frac{2\alpha_t^q}{q} + \frac{2Y_t^{2p}}{p} + \frac{2\beta_t^q}{q} + \frac{2Y_t^{2p}}{p\lambda} + \frac{2\gamma_t^{2q}}{q\lambda} + \frac{\lambda Z_t^2}{2} + \frac{2Y_t^{2p}|\ln(Y_t)|^p}{p}\right) dt\Bigg], \end{aligned}$$

where we have used Young's inequality. Then, the thesis follows arguing as above by choosing $\lambda = |2\delta - 1|$. The case $\delta = 1/2$ can be shown similarly, employing the substitution $f(x) = x^2 \ln(x)$. □

We need a preliminary lemma, provided in [2]. For the reader's convenience, we state this result again.

Lemma A.2.1. *Let $g, g_1, g_2 : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ be $\mathcal{P} \times \mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R}^n)/\mathcal{B}(\mathbb{R})$-measurable functions and $X, X_1, X_2 \in L^0(\mathcal{F}_T)$ with $X_1 \leq X \leq X_2$. Let us assume that the BSDEs with parameters (X_1, g_1) and (X_2, g_2) admit solutions (Y^1, Z^1) and (Y^2, Z^2), respectively, such that:*

- *$Y^1 \leq Y^2$,*
- *$d\mathbb{P} \times dt$-a.s., $y \in \left[Y_t^1(\omega), Y_t^2(\omega)\right]$ and $z \in \mathbb{R}^n$ it holds that: $g_1(t, y, z) \leq g(t, y, z) \leq g_2(t, y, z)$ and $|g(\omega, t, y, z)| \leq \eta_t(\omega) + C_t(\omega)|z|^2$, where C and η are pathwise continuous and $(\mathcal{F}_t)_{t\in[0,T]}$-adapted processes with η verifying $\int_0^T |\eta_s| ds < +\infty$ $d\mathbb{P}$-a.s.*

Then, if g is continuous in (y, z) $d\mathbb{P} \times dt$-a.s., the BSDE with parameters (X, g) admits at least one solution (Y, Z) in the sense of Definition 4, with $Y^1 \leq Y \leq Y^2$. In addition, among all solutions lying between Y^1 and Y^2, there exist a maximal and a minimal solution.

Proof of Corollary 12. With the same notation as in Proposition 10, we know that the BSDE with parameters $(\bar{X}, \bar{h})$ admits a unique solution in $\mathcal{H}_T^{e^{\lambda T}+1} \times \mathcal{M}_T^2$, according to Theorem 2.1 in [5]. Hence, $Y \in \mathcal{H}_T^{(2\delta+1)(e^{\lambda T}+1)}$, while the regularity of Z can be checked as in the proof of Proposition 10. Thus, considering two solutions (Y, Z) and (U, Q) to the BSDE with parameter (X, h), we have $\bar{Y} = Y^{2\delta+1} = U^{2\delta+1} = \bar{U} \in \mathcal{H}_T^{(2\delta+1)(e^{\lambda T}+1)}$ and $\bar{Z} = (2\delta+1)Y^{2\delta}Z = (2\delta+1)U^{2\delta}Q = \bar{Q} \in \mathcal{M}_T^2$. In particular, $U = Q$ $dt \times d\mathbb{P}$-a.s. since

$$\begin{aligned}\int_0^T |Z_t - Q_t|^2 dt &= \int_0^T Y_t^{-4\delta} |Y_t^{2\delta} Z_t - U_t^{2\delta} Q_t|^2 dt \\ &\leq \operatorname*{ess\,sup}_{t\in[0,T]} \{Y_t^{-4\delta}\} \int_0^T |Y_t^{2\delta} Z_t - U_t^{2\delta} Q_t|^2 dt \\ &\leq C_\omega \int_0^T |\bar{Z}_t - \bar{Q}_t|^2 dt = 0 \ d\mathbb{P}\text{-a.s.},\end{aligned}$$

where the equality $Y = U$ leads to the above equality, while the last inequality is due to the pathwise continuity of Y, $Y > 0$ and $\bar{Z} = \bar{Q}$. □

Proof of Proposition 14. The existence and uniqueness of the solution to Equation (4.4) has already been established in Corollary 12. Analogously, Theorem 11 guarantees the existence of a minimal and a maximal solution to the BSDE with parameters (X, g). We need to check the regularity of these solutions.

Boundedness of Y for Equation (4.4): It suffices to show that $\bar{Y}^h := (Y^h)^{2\delta+1}$ is bounded, as this implies the boundedness of Y^h. We can prove that $\bar{Y}^h$ is bounded by employing the same transformation as in Proposition 10 and applying Corollary 3.4 (i) in [2].

Boundedness of Y for a general driver g: By Theorem 11, we know that the Y-component of the solution to the BSDE with parameters (X, g) verifies $0 < Y \leq Y^h$, hence boundedness of Y^h implies the same property for Y.

$Z \in BMO(\mathbb{P})$: In order to prove that $Z \in \mathrm{BMO}(\mathbb{P})$ we first show that $Z \in \mathcal{M}_T^2$. Proceeding as in the proof of Theorem 11, applying Itô's formula to $f(x) := \ln(1+x)$, it holds that

$$\frac{1}{2(1+\|Y\|_\infty^T)^2} \mathbb{E}\left[\int_0^T |Z_s|^2 ds\right] \leq \mathbb{E}\left[\int_0^T \frac{1}{2(1+Y_s)^2} |Z_s|^2 ds\right] \leq Y_0 + \|X\|_\infty < +\infty,$$

where we employed boundedness of X. Hence, $Z \in \mathcal{M}_T^2$. Having proved that $Z \in \mathcal{M}_T^2$ and Y is bounded and positive, the process $\frac{Z}{1+Y} \in \mathcal{M}_T^2$. Thus, we can follow the passage as above, without incorporating the localization procedure, as the stochastic integral of $\frac{1}{(1+Y)^2}|Z|^2$ is a uniformly integrable martingale. Consequently, we have

$$\begin{aligned}&\frac{1}{2(1+\|Y\|_\infty^T)^2} \mathbb{E}\left[\int_t^T |Z_s|^2 ds \middle| \mathcal{F}_t\right] \leq \mathbb{E}\left[\int_t^T \frac{1}{2(1+Y_s)^2} |Z_s|^2 ds \middle| \mathcal{F}_t\right] \\ &\leq \|Y\|_\infty^T \leq C < +\infty,\end{aligned}$$

thus $Z \in \mathrm{BMO}(\mathbb{P})$. □

Proof of Corollary 15. The BSDE

$$Y_t^h = X + \int_t^T \left(\alpha_s Y_s^h + \beta_s Y_s^h |\ln(Y_s^h)| + \gamma_s \cdot Z_s^h + \delta |Z_s^h|^2 / Y_s^h\right) ds - \int_t^T Z_s^h dW_s$$

can be rewritten as

$$Y_t^h = X + \int_t^T \left(\alpha_s Y_s^h + \beta_s Y_s^h |\ln(Y_s^h)| + \delta |Z_s^h|^2 / Y_s^h\right) ds - \int_t^T Z_s^h dW_s^\gamma, \tag{A.IV}$$

where Girsanov's theorem is employed. Then, by Proposition 14, there exists a (unique) solution (Y^h, Z^h) to Equation (A.IV) such that $|Y^h_t| \leq K\ d\mathbb{Q}^\gamma \times dt$-a.s. and $Z^h \in \mathrm{BMO}(\mathbb{Q}^\gamma)$. Here, $\mathbb{Q}^\gamma$ is the probability measure with density $\mathcal{E}^\gamma_T$. As is well known, $\mathbb{Q}^\gamma$-boundedness implies $\mathbb{P}$-boundedness. Furthermore, any $\mathrm{BMO}(\mathbb{Q}^\gamma)$ martingale is also a $\mathrm{BMO}(\mathbb{P})$ martingale (see for instance [7]). Hence, Equation (A.IV) admits a unique solution $(Y^h, Z^h) \in \mathcal{H}^\infty_T \times \mathrm{BMO}(\mathbb{P})$. The existence of a solution (Y, Z) with the required regularity for the general BSDE with parameters (X, g) follows similarly as in the proofs of Theorem 11 and Proposition 14. □

Proof of Lemma 16. We want to apply Theorem 1 in [33] to the function

$$\bar{\psi}(x) = \begin{cases} 2\ln(2) & \text{if } 0 \leq x \leq 2, \\ x\ln(x) & \text{if } x > 2. \end{cases}$$

It is clear that $\bar{\psi}$ is convex, positive and non-decreasing on $\mathbb{R}_+$. Furthermore, we have that $x|\ln(x)| \leq \bar{\psi}(x)$ for any $x \in \mathbb{R}_+$ and $\psi(x) = \int_2^x \frac{1}{\bar{\psi}(r)}dr$, for any $x > 0$. The integrability conditions on X, β, u guarantee that $\mathbb{E}[|X + \int_0^T \beta_s \bar{\psi}(u_s)ds|^{p'e^B}] < +\infty$, for some $p' > 1$. Indeed, $\bar{\psi}(x) \leq C_\varepsilon + x^\varepsilon$ for any $\varepsilon > 1$, with $C_\varepsilon > 0$. Thus, it holds that

$$\mathbb{E}\left[X + \int_0^T \beta_s \bar{\psi}(u_s)ds\right] \leq C_\varepsilon + \mathbb{E}\left[X + \int_0^T \beta_s u_s^\varepsilon ds\right] \leq C_\varepsilon + \mathbb{E}\left[X + \|\beta\|_\infty^T \operatorname*{ess\,sup}_{t\in[0,T]} u_s^\varepsilon\right].$$

Choosing $\varepsilon > 1$ such that $p'e^B = \varepsilon p e^B = p(e^B + 1)$ (hence, $\varepsilon = 1 + e^{-B}$) with $p' = \varepsilon p > 1$, the integrability follows. Thus, we only need to check that

$$\psi^{-1}\left(\psi\left(\mathbb{E}\left[X + \int_t^T \beta_s \bar{\psi}(u_s)ds\right]\right) + \int_0^t \beta_s ds\right)$$

is uniformly integrable in time. Setting $X_t := \mathbb{E}\left[X + \int_t^T \beta_s \bar{\psi}(u_s)ds \middle| \mathcal{F}_t\right]$ it holds by direct inspection that

$$\begin{aligned} &\psi^{-1}\left(\psi(X_t) + \int_0^t \beta_s ds\right) = \exp\left(\exp(\ln(\ln(X_t)) + \int_0^t \beta_s ds + \ln(\ln(2)))\right) \\ &\leq \exp(e^B \ln(X_t))) = (X_t)^{e^B} = \mathbb{E}\left[X + \int_t^T \beta_s \bar{\psi}(u_s)ds \middle| \mathcal{F}_t\right]^{e^B} \\ &\leq \mathbb{E}\left[X + \int_0^T \beta_s \bar{\psi}(u_s)ds \middle| \mathcal{F}_t\right]^{e^B}. \end{aligned}$$

The integrability conditions on X, β, u and the properties of conditional expectations ensure that

$$\mathbb{E}\left[\left|\mathbb{E}\left[X + \int_0^T \beta_s \bar{\psi}(u_s)ds \middle| \mathcal{F}_t\right]\right|^{p'e^B}\right] \leq \mathbb{E}\left[\left|X + \int_0^T \beta_s \bar{\psi}(u_s)ds\right|^{p'e^B}\right] < +\infty,$$

thus the process $\psi^{-1}(\psi(X_t) + \int_0^t \beta_s ds)$ is bounded in $L^{p'}$, $p' > 1$. Hence, it is uniformly integrable and the thesis follows. □

Proof of Corollary 21. As is clear from Theorem 20, we only need to verify the convergence $\int_0^T g^n(t, Y_t, Z_t)dt \to \int_0^T g(t, Y_t, Z_t)dt$ in L^p for any $p \geq 1$ to establish the thesis. For any fixed $p \geq 1$, it holds that

$$\begin{aligned} &\mathbb{E}\left[\left|\int_0^T g^n(t, Y_t, Z_t)dt\right|^p\right] \leq \mathbb{E}\left[\left(\int_0^T |g^n(t, Y_t, Z_t)|dt\right)^p\right] \\ &\leq \mathbb{E}\left[\left(\int_0^T \left(\alpha_t Y_t + \beta_t Y_t|\ln(Y_t)| + \gamma_t|Z_t| + \delta\frac{|Z_t|^2}{Y_t}\right) dt\right)^p\right]. \end{aligned}$$

Then, by the assumptions $X \in L^p$ for any $p \geq 1$, $X \geq \varepsilon$ and applying Proposition 19, we obtain uniform integrability of $\int_0^T g^n(t, Y_t, Z_t)dt$ for any $p \geq 1$. In addition, $g^n(t, V_t, Z_t) \xrightarrow{n\to\infty} g(t, V_t, Z_t)$ $d\mathbb{P} \times dt$-a.s., thus Vitali's theorem leads to the thesis. □

Proof of Proposition 28. Consider two solutions (Y, Z) and (Y', Z') to Equation (5.1) such that $0 < Y, Y' \leq Y^{h_1,h_2}$. Then the corresponding one-driver equation with parameters (X, g) with $g(t, y, v) := g_1(t, y, g_2^{-1}(t, y, v))$ verifies all the assumptions of Theorem 17, hence there exists a unique solution (Y^g, Z^g) to the BSDE with parameters (X, g). Clearly, this implies the uniqueness of the first component of the solution to Equation (5.1). We only need to verify the uniqueness for the z-component. Proceeding exactly as in the proof of Theorem 17, we can prove that

$$\int_0^T |g_2(t, Y_t, Z_t) - g_2(t, Y_t, Z'_t)|^2 = 0 \;\text{ a.s.},$$

which entails

$$g_2(t, Y_t, Z_t) = g_2(t, Y_t, Z'_t), \quad d\mathbb{P} \times dt\text{-a.s.}$$

Injectivity of g_2 w.r.t. z gives $Z_t = Z'_t$ $d\mathbb{P} \times dt$-a.s. □

Proof of Corollary 30. Let us observe that Equation (3.4) can be rewritten as a two-driver BSDE, by defining $g_1(t, y, z) := y\tilde{f}(t, y, z)$ and $g_2(t, y, z) := yz$. Hence, the part of the statement regarding existence is a straightforward consequence of Proposition 26. For the uniqueness, we have that $g_1(t, y, g_2^{-1}(t, y, v)) = y\tilde{f}(t, y, v/y) = y\tilde{f}_1(t, y) + y\tilde{f}_2(t, v/y)$. Thus, $y \mapsto y\tilde{f}_1(\cdot, y)$ is convex by assumption and $(y, v) \mapsto y\tilde{f}_2(v/y)$ is jointly convex in (y, v) being the perspective function of the convex function $\tilde{f}_2$. Hence, $g_1(t, y, g_2^{-1}(t, y, v))$ is jointly convex in (y, v) and uniqueness follows from Proposition 28. The last thesis is obvious, by taking $\tilde{f}_1 \equiv 0$. □