Optimal Savings and Portfolio Choice with Risky Labor Income and Loss Aversion*

Servaas van Bilsen
Amsterdam School of Economics
University of Amsterdam
and NETSPAR

Roger J. A. Laeven†
Amsterdam School of Economics
University of Amsterdam
EURANDOM and CentER

Theo E. Nijman
Department of Finance
Tilburg University
CentER and NETSPAR

February 2, 2026

Abstract

This paper derives and explores the savings and portfolio decisions of a loss-averse individual who faces risky labor income. We show that her savings rate and portfolio share are more negatively affected by income declines than under conventional preferences: a loss-averse individual strongly guards consumption by reducing savings, conducts excessive de-risking, and often prefers to withdraw wealth prior to retirement. The optimal responses exhibit large heterogeneity across consumption-to-reference level ratios. We also find that the investment strategy is conservative when old and aggressive at younger ages, compared to standard preferences. Welfare losses associated with implementing conventional plans easily exceed 25%.

JEL Classification: D81, G11, J24, J26.

Keywords: Risky Labor Income; Loss Aversion; Endogenous Reference Level; Optimal Savings Behavior; Optimal Portfolio Behavior.

---

*We are very grateful to Yacine Aït-Sahalia, Lans Bovenberg, Jeffrey Brown, Louis Eeckhoudt, Frank de Jong, Leonid Kogan, Pierre-Carl Michaud, Olivia S. Mitchell, Jonathan Parker, Richard Peter (discussant), Hans Schumacher, and to conference and seminar participants at the Australasian Finance and Banking Conference, the 30th CEPAR/UNSW Colloquium on Pensions and Retirement Research, the CORS/INFORMS International Conference, the European Winter Meetings of the Econometric Society, the 50th Seminar of the European Group of Risk and Insurance Economists, Heriot-Watt University, the Insurance, Mathematics and Economics International Congress, the Netspar International Pension Workshop, the Netspar Pension Day, Sun Yat-Sen University, Tilburg University, the Tinbergen Institute, the University of Amsterdam and the University of New South Wales for their helpful comments and suggestions. This paper was circulated earlier under the title "Optimal Savings and Portfolio Choice with Risky Labor Income and Reference-Dependent Preferences". This research was supported in part by the Netherlands Organization for Scientific Research under an NWO-Vici grant 2020–2027 (Laeven).
Email addresses: `S.vanBilsen@uva.nl`; `R.J.A.Laeven@uva.nl`; `Nyman@uvt.nl`.

†Corresponding author. Mailing Address: PO Box 15867, 1001 NJ Amsterdam, The Netherlands. Phone: +31 (0) 20 525 4252.

1 Introduction

Labor income is the most important asset for many households. Indeed, as estimated by The World Bank Group (2021), human wealth, which equals the present value of future labor income, constitutes 71.2% of total US wealth.[1] Yet, labor income is far from riskless, as has been amply demonstrated by the recent COVID-19 crisis.[2] Hence, an important question is how risky labor income affects individuals' optimal savings and portfolio decisions. We analyze this question for an individual with modern preferences featuring loss aversion and reference dependence, which are prominent behavioral regularities documented in a large experimental and empirical literature.

We show that the negative impact of a labor income decline on both the optimal savings rate and the optimal risky portfolio share is significantly more pronounced under loss-averse reference-dependent preferences than under conventional, non-behavioral preferences.[3] Our findings reveal a notable behavioral opposition: when confronted with a labor income shock, loss-averse individuals tend to preserve current consumption by strongly reducing savings and excessively de-risk their portfolios – a response that contrasts sharply with the familiar prediction of increased saving and additional risk-taking following a negative stock return shock.[4] Hence, the direction and magnitude of the optimal responses of the savings rate and risky portfolio share depend critically on the source of risk. In many cases, our individuals even exhibit a preference

[1] See also Mayers (1972), Jorgenson and Fraumeni (1989) and Lustig, van Nieuwerburgh, and Verdelhan (2013).

[2] The unemployment rate in the US increased from 3.5% in February 2020 to 14.7% in April 2020. In August 2021, the unemployment rate was 5.2%, which is still considerably higher than in the three years prior to the COVID-19 crisis (US Bureau of Labor Statistics). In the EU, total employment income decreased by 4.8% in 2020, mainly due to absences and reduced hours (EUROSTAT).

[3] Throughout, by conventional (or standard) preferences, we mean CRRA (i.e., iso-elastic, power utility) expected utility preferences, unless stated otherwise. Section 1.1 reviews the related literature, including papers with labor income risk, and discusses how our paper differs from this literature.

[4] This familiar prediction occurs already under standard preferences, a finding that can be traced back to Merton (1969), and even more so under behavioral preferences (e.g., Van Bilsen, Laeven, and Nijman (2020b)).

to withdraw financial wealth prior to retirement, reflecting a strong aversion to consumption losses relative to their reference levels. These effects are particularly pronounced among individuals with low consumption-to-reference level ratios, who face greater difficulty adjusting their current consumption in response to income shocks. By contrast, individuals with high consumption-to-reference level ratios are generally less constrained and can more easily accommodate reductions in income without sharply reducing savings or risk-taking. Furthermore, we find that the optimal investment strategy under loss aversion tends to be more conservative at older ages and more aggressive at younger ages, relative to strategies derived from conventional preferences that incorporate risky labor income. Importantly, welfare losses resulting from applying standard, non-behavioral savings and investment policies can easily exceed 25%, underscoring the substantial costs of ignoring reference dependence and loss aversion in financial life-cycle planning. Our findings remain qualitatively robust across a wide range of parameter values.

In our analysis, we consider an individual who decides at every time instance how much to save and how to invest. She draws utility from the difference between consumption and a reference level, in line with the seminal work of Kahneman and Tversky (1979) and Tversky and Kahneman (1992). Negative deviations in consumption from the reference level loom larger than positive deviations. That is, our individual is loss averse. Furthermore, the reference level is endogenous and depends upon own past optimal consumption decisions.[5] In addition to stock market risk, the individual faces both permanent and transitory labor income risk. Shocks to labor income cannot be hedged, thus exposing the individual to an important, non-tradable source of risk, which substantially complicates the analysis of the optimal policies.

We explicitly derive and explore the optimal savings and portfolio strategies. Our

[5] In an extension of the base model, the individual's reference level also depends on own past labor income and the past consumption choices of the individual's neighbors, in line with, e.g., Langtry (2023).

analysis yields three main findings. First, we find that both the optimal savings rate and the share of wealth allocated to risky assets are more adversely affected by a decline in labor income when individuals exhibit loss-averse reference-dependent preferences, compared to those with conventional preferences. Under these standard preferences, a negative labor income shock already leads to a reduction in the optimal savings rate.[6] Under loss-averse reference-dependent preferences, this response becomes further amplified. The endogeneity of the reference level and the desire to avoid losses in current consumption generate excess sensitivity: a strong downward adjustment in savings and risk-taking following a labor income decline. A loss-averse individual with an endogenous reference level strongly prefers to shield current consumption. As a result, she absorbs a drop in labor income by strongly cutting back on savings and excessively reducing the exposure to risky assets. By contrast, the response to a negative stock return shock is markedly different. In that case, the individual increases her savings rate and allocates a larger share of her portfolio to risky assets. This behavior reflects an attempt to recover from financial losses. The opposite responses – heavily reducing savings and risk-taking after a labor income shock versus increasing savings and risk-taking after a stock return shock – is a distinctive implication of our setting with loss-averse reference-dependent preferences and risky labor income.

We even find that, in the case of a substantial drop in labor income, the individual does not save at all and already withdraws financial wealth prior to retirement. This behavior underscores the potential downside of institutional constraints that limit early access to retirement savings. In particular, settings where individuals are unable to unlock financial wealth before retirement can lead to significant welfare losses. For example, in the United States, individuals are generally not permitted to withdraw funds from tax-advantaged

[6]Under standard preferences, a labor income shock affects consumption not only during the working phase but also during the retirement phase. Accordingly, following a drop in labor income, the individual reduces her savings rate, leading to lower financial wealth available for retirement consumption.

accounts such as 401(k) plans or Individual Retirement Accounts (IRAs) before the age of 60. While these restrictions are intended to preserve retirement security, they may be counterproductive in circumstances such as job loss, where early access to savings could smooth consumption more efficiently. Our analysis shows that the welfare loss associated with conventional preferences – under which early withdrawal of financial wealth is less likely than under loss-averse reference-dependent preferences – can reach up to 25%.

Furthermore, we can explain the excess sensitivity of the optimal risky portfolio share using a time-varying relative risk aversion. Following a decline in labor income, the individual reduces her optimal share of risky assets not only because her expected future labor income falls – thereby weakening the diversification benefits of labor income against stock return risk – but also because she typically becomes more risk averse. Indeed, after an income shock, the individual becomes more concerned about the possibility that future consumption might fall below her reference level. To avoid such consumption losses, she chooses to de-risk: shifting her investments away from risky stocks toward safer, riskless assets.

Our second main finding is that the optimal response of both the savings rate and the risky portfolio share to a labor income shock varies significantly with the ratio of consumption to the reference level. When consumption is close to the reference level, the individual responds to a drop in labor income by drastically reducing both her savings rate and her allocation to risky assets. Intuitively, as consumption approaches the reference level, the individual becomes increasingly concerned about potential consumption losses relative to the reference level and thus prefers to strongly protect current consumption. To do so, she lowers her savings rate and reduces the exposure to stock market risk. By contrast, when consumption is far above the reference level, a decline in labor income has a much smaller effect on savings behavior and portfolio choice. Importantly, this behavior differs markedly from the case of conventional preferences, under which the optimal responses of the savings rate and risky portfolio share are independent of the ratio between consumption and the reference level.

Our third main finding is that the optimal portfolio strategy is typically more conservative at older ages and more aggressive at younger ages, compared to the case with standard preferences and risky labor income. This age-dependent pattern arises from the interplay between risky labor income and loss aversion with an endogenous reference level, both of which shape the individual's risk-taking behavior over the life cycle. Consistent with existing literature (e.g., Viceira (2001)), risky labor income reduces the optimal share of financial wealth allocated to risky assets, as it increases overall income risk and lowers the diversification benefits of labor income against stock return risk. However, the introduction of loss aversion with an endogenous reference level generates two counteracting effects on the optimal portfolio choice. On the one hand, the endogeneity of the reference level encourages riskier investment behavior. Since individuals gradually adjust their consumption over time in response to shocks, they can limit the impact of a shock on current consumption. This adaptive behavior makes them more willing to take stock return risk, particularly at younger ages, when there is more time to recover from negative shocks. On the other hand, loss aversion implies a strong preference to avoid future consumption falling below the reference level, prompting a more conservative investment strategy to minimize the risk of consumption losses. This precautionary motive becomes increasingly important with age, as there is less time to recover from shocks. Overall, for a wide and realistic range of parameter values, we find that the optimal portfolio strategy under loss-averse reference-dependent preferences is more conservative at older ages and more aggressive at younger ages, diverging from predictions under standard preference models with risky labor income.

An important methodological contribution of our work is the development of a novel solution approach tailored to the challenges posed by market incompleteness and loss-averse reference-dependent preferences. Standard solution methods are not applicable in this setting. We therefore construct a non-trivial solution technique that allows us to explicitly derive the optimal savings and portfolio policies, and we develop a comprehensive numerical algorithm to compute the shadow price of labor income risk. This shadow price plays a central role in obtaining explicit solutions, making it a valuable tool for both theoretical analysis and intuitive interpretation of the optimal strategies. Our approach extends and adapts the frameworks of He and Pearson (1991), Schroder and Skiadas (2002), and Van Bilsen et al. (2020b) to accommodate the presence of behavioral preferences and non-tradable labor income risk.

1.1 Related Literature

The question of how to optimally save and invest total wealth over the life-cycle has been extensively studied in different contexts since the seminal works of Mossin (1968), Merton (1969) and Samuelson (1969).[7] Bodie, Merton, and Samuelson (1992) was the first to include human wealth into the analysis and shows that in a setting in which labor income is riskless and CRRA preferences apply, the optimal share of financial wealth invested in the risky stock decreases, on average, with age. Intuitively, the bond-like asset human wealth diversifies stock return risk. Hence, early in life, when human wealth is large, the individual can afford to take more stock return risk. Furthermore, they show that the optimal consumption drawdown strategy remains state-independent.

A number of authors (see, e.g., Viceira (2001), Cocco, Gomes, and Maenhout (2005), Benzoni, Collin-Dufresne, and Goldstein (2007), and Schlafmann, Setty, and Vestman (2024)) explore optimal household behavior with (non-tradable) *risky* labor income.[8]

[7] For example, Campbell, Cocco, Gomes, Maenhout, and Viceira (2001), Wachter (2002), Chacko and Viceira (2005), Liu (2007), and Laeven and Stadje (2014) among other papers study how stochastic rather than constant investment opportunities affect household consumption and portfolio decisions.

[8] See also, e.g., Gomes and Michaelides (2003), Chai, Horneff, Maurer, and Mitchell (2011), Hubener, Maurer, and Mitchell (2016), and Maurer, Mitchell, Rogalla, and Schimetschek (2021).

These papers are successful in explaining several empirical facts such as most noticeably the hump-shaped equity allocation over the life-cycle (as documented, e.g., by Ameriks and Zeldes (2004)). Most papers in this literature describe household preferences by either CRRA utility or Epstein-Zin utility.

The experimental and empirical literature has, however, shown substantial deviations from these conventional preferences. One prominent finding is that people tend to evaluate outcomes relative to a reference level; see the classical works of Kahneman and Tversky (1979) and Tversky and Kahneman (1992). Several papers explore the savings and investment implications of loss-averse reference-dependent preferences (see, e.g., Berkelaar, Kouwenberg, and Post (2004), Gomes (2005), and Van Bilsen et al. (2020b)).[9,10] Most of these authors do not consider labor income risk. We ask how uncertainty in labor income affects the optimal savings and portfolio strategies in this behavioral environment.

Our paper is perhaps most closely related to the work by Pagel (2018), which also incorporates loss aversion in consumption and allows for risky labor income.[11] While both papers share these features, the focus and approach of Pagel (2018) differ from ours in a number of ways. First, Pagel (2018) focuses on inattention and delegation in portfolio choice. Second, under the preferences of Pagel (2018), the individual experiences loss-averse utility over news and the reference level is based on expectations, whereas in our setting the reference level depends on past consumption experiences. Third, to derive actual optimal strategies, Pagel (2018) relies on numerical backward induction.[12] Our

[9]See also, e.g., Jin and Zhou (2008), He and Zhou (2011), He and Zhou (2016), Curatola (2017), Guasoni, Huberman, and Ren (2020),Van Bilsen, Bovenberg, and Laeven (2020a), Deng, Li, Pham, and Yu (2022), He and Strub (2022), Liang, Luo, and Yuan (2023), Liu, Yu, and Zhang (2024), and Wang, Liu, and Siu (2024).

[10]This paper considers a partial equilibrium model; that is, we assume that asset prices are exogenously given. For the asset-pricing implications of loss aversion, see, e.g., Shumway (1997), Barberis, Huang, and Santos (2001), Yogo (2008), Andries (2012), and Pagel (2016).

[11]Building on the work of Kőszegi and Rabin (2006), Pagel (2017) studies optimal consumption decisions in the presence of risky labor income but does not consider optimal portfolio choice.

[12]See Appendix D.2 in Pagel (2018).

paper develops a novel solution method that allows us to derive the optimal policies analytically, up to the numerical determination of the shadow price of labor income risk. This enables us to characterize and provide intuitive interpretations to the interaction between labor income risk and consumption and investment behavior in a behavioral setting.

Finally, we note that a backward-looking rule for updating the reference level has been widely employed in the habit formation literature since Constantinides (1990). Within that literature, individuals are not loss averse over consumption, since the reference level acts as a hard floor and income is implicitly assumed to be sufficient to guarantee consumption at that level. By contrast, our paper adopts a more flexible approach by interpreting the reference level as a standard of living. In our setting, individuals can experience consumption shortfalls relative to this reference level, and they are explicitly loss averse over such losses. Moreover, income is uncertain and, by construction, there is no guaranteed floor, so individuals face the risk of both income and consumption falling below their standard of living – sharply contrasting with the assumptions and implications of the standard habit formation literature.

The remainder is structured as follows. Section 2 introduces the model specification, while Section 3 presents the optimal policies and outlines the solution technique. The main findings are discussed in Section 4. This section also indicates how our findings can be tested using household data. Finally, Section 5 provides a conclusion. The proofs and several technical details are relegated to Appendices A and B. We describe the numerical algorithm in an online appendix.

2 Model

This section presents our continuous-time model. Denote by t adult age, which corresponds to the individual's chronological age minus the age at which she starts working. To focus on the impact of risky labor income on the individual's optimal savings and investment strategies, we abstract away from longevity risk, assuming that the individual retires at the (fixed) adult age $T_R > 0$ and dies at the (fixed) adult age $T_D > 0$.

2.1 State Variables, Labor Income and Financial Market

We consider an economy with two state variables: risky labor income $Y(t)$ and the stock price $S(t)$. We assume rather generic dynamics for individual labor income. We generalize the labor income model considered by Wang, Wang, and Yang (2016) and specify individual labor income $Y(t)$ as follows ($t \leq T_R$):[13]

$$Y(t) = Y(0) \exp\left\{ \int_0^t \mu_Y(s)\mathrm{d}s + \sigma_Y \int_0^t e^{-\kappa(t-s)}\mathrm{d}Z_Y(s) \right\}, \tag{2.1}$$

$Y(0) > 0$. Here, $\mu_Y(t) \in \mathbb{R}$ represents the growth rate of median labor income, $\sigma_Y \geq 0$ denotes (instantaneous) labor income volatility, $Z_Y(t)$ is a standard Brownian motion, and $\kappa \geq 0$ models the impact of the labor income shock $\sigma_Y \mathrm{d}Z_Y(t)$ on future median labor income. We allow κ to be different from zero to capture the idea that labor income shocks can be permanent or transitory; see, e.g., Caroll (1997). If $\kappa = 0$, then labor income shocks are permanent, while if $\kappa = \infty$, then labor income shocks are fully transitory, i.e., labor income shocks will not affect future median labor income. Figure 12 in Section 4.4 illustrates the impact of a labor income shock on future median labor income for various values of κ. Moreover, we allow the growth rate of median labor income $\mu_Y(t)$ to be

[13] If $\kappa = 0$ and $\mu_Y(s)$ is age-independent, (2.1) reduces to the model considered by Wang et al. (2016).

time-dependent, to capture the idea that the drift of the labor income process typically depends on age; see, e.g., Cocco et al. (2005). Finally, we note that, after retirement, $Y(t)$ represents the social security payment from the government at adult age t.[14]

We assume that the individual can invest her financial (pension) wealth $F(t)$ into two assets: a risky stock and a riskless asset. We assume the following dynamics for the stock price $S(t)$ and the price of the riskless asset $B(t)$:

$$\mathrm{d}S(t) = (r + \lambda_S \sigma_S)\, S(t)\mathrm{d}t + \sigma_S S(t)\mathrm{d}Z_S(t), \tag{2.2}$$

$$\mathrm{d}B(t) = rB(t)\mathrm{d}t, \tag{2.3}$$

$S(0), B(0) > 0$. Here, $\lambda_S \in \mathbb{R}$ denotes the market price of stock return risk, $\sigma_S > 0$ models the stock return volatility, $Z_S(t)$ is a standard Brownian motion, and $r \in \mathbb{R}$ denotes the riskless interest rate. We allow the Brownian motions $Z_S(t)$ and $Z_Y(t)$ to be correlated, and denote the associated correlation coefficient by $\rho_{SY} \in [-1, +1]$.

2.2 Preferences

Denote by $c(t)$ and $h(t)$ the individual's consumption choice and the individual's endogenous reference level at adult age t, respectively. We assume that the individual derives utility from the difference between consumption and an endogenously-updated consumption reference level. More specifically, expected lifetime utility is given by

$$U = \mathbb{E}_0 \left[\int_0^{T_D} e^{-\delta t} u\,(c(t) - h(t))\,\mathrm{d}t \right], \tag{2.4}$$

[14] We assume that the size of the social security payment is determined at the retirement age T_R and is a function of the history of individual labor income $Y(s)$, $s \in [0, T_R]$. In what follows, we refer to $Y(t)$ simply as individual labor income at t.

with[15]

$$u\left(c(t)-h(t)\right)=\begin{cases}\left(c(t)-h(t)\right)^{\gamma_G}, & \text{if } c(t)\geq h(t);\\ -\eta\left(h(t)-c(t)\right)^{\gamma_L}, & \text{if } c(t)<h(t).\end{cases} \tag{2.5}$$

Here, $\mathbb{E}_0[\cdot]$ represents the expectation conditional upon information available at time 0. The instantaneous utility function $u(\cdot)$ takes the form of the familiar two-part power utility function under which risk preferences for gains are different than risk preferences for losses and one dollar loss looms larger than one dollar gain. The parameters $\gamma_G \in (0,1)$ and $\gamma_L > 0$ dictate the curvature for gains and losses, respectively, $\eta \geq 1$ stands for the loss aversion parameter,[16] and $\delta \geq 0$ denotes the subjective rate of time preference. The individual is risk averse in the gain domain and either risk loving ($0 < \gamma_L \leq 1$) or risk averse ($\gamma_L > 1$) in the loss domain.[17]

We assume that the reference level satisfies the dynamics:

$$\mathrm{d}h(t) = \left(\beta c(t) - \alpha(t)h(t)\right)\mathrm{d}t, \tag{2.6}$$

$h(0) > 0$. Specification (2.6) is our base specification.[18] We allow the depreciation

[15]We consider loss aversion in consumption rather than in wealth. Consumption-based loss aversion is particularly appropriate for household life-cycle planning, as households are sensitive to changes in their consumption patterns, i.e., standard of living. By contrast, wealth-based loss aversion is more relevant for analyzing pure investment behavior. For example, Barberis, Huang, and Thaler (2006) study how the interaction of narrow framing and loss aversion can lead investors to reject favorable but risky investments, thereby helping to explain limited stock market participation.

[16]There is ongoing debate in the literature regarding the precise definition of loss aversion. In this paper, we adopt the formulation of Tversky and Kahneman (1992). According to their formulation, the disutility associated with a unit loss is η times greater than the utility derived from a unit gain. From Section 4, we assume that $\gamma_L > \gamma_G$, which jointly with $\eta > 1$ ensures that losses loom larger than equivalent gains, i.e., $-u(-x) > u(x)$ for all $x > 1$. This assumption is particularly plausible in a life-cycle context, where the stakes involved are typically substantial.

[17]In the special case in which losses are excluded, i.e., $\eta = \infty$, the curvature parameter for gains γ_G is allowed to be larger than 1, i.e., $\gamma_G \in (0, \infty)$.

[18]In the appendix (and the online appendix), we consider an even more general reference level specification that depends not only on own past consumption but also on individual past labor income and on past consumption of the individual's neighbors. A reference level specification in which the reference level depends on other people's consumption captures the idea that the individual wants to

parameter $\alpha(t)$ to be a deterministic function of adult age t, so that, for example, the rate of depreciation during the working phase can differ from the rate of depreciation during the retirement phase. The preference parameter $\beta \geq 0$ models the impact of current consumption $c(t)$ on the individual's current consumption reference level $h(t)$. Our reference level specification (2.6) is in line with, or nests as special cases, the specifications considered by, e.g., Constantinides (1990), Detemple and Zapatero (1991), Gomes and Michaelides (2003), Munk (2008), and Van Bilsen et al. (2020b).

2.3 Dynamic Budget Constraint

The individual invests all her financial wealth, which includes (past) labor income that is saved and not consumed. Let us denote by $\omega(t)$ the share of financial wealth $F(t)$ invested in the risky stock at adult age t. The individual's dynamic budget constraint is now given by

$$\mathrm{d}F(t) = (r + \omega(t)\lambda_S\sigma_S)\, F(t)\mathrm{d}t + \omega(t)\sigma_S F(t)\mathrm{d}Z_S(t) + (Y(t) - c(t))\mathrm{d}t, \tag{2.7}$$

$F(0) > 0$. We observe from (2.7) that financial wealth grows because of two reasons: investment results (see the first two terms on the right-hand side of (2.7)) and new savings $Y(t) - c(t)$. Admissible consumption and investment strategies ensure that $F(t)$ is bounded from below (Karatzas and Shreve (1998), pp. 91–92).

catch up with the Joneses; see, e.g., Abel (1990), Campbell and Cochrane (1999), Chen (2017), and Langtry (2023).

2.4 Dynamic Maximization Problem

The individual faces the following dynamic maximization problem:

$$
\begin{aligned}
&\max_{c(t),\omega(t)} \quad \mathbb{E}_0\left[\int_0^{T_D} e^{-\delta t} u\left(c(t)-h(t)\right) \mathrm{d}t\right] \\
&\qquad\qquad \mathrm{d}h(t) = \left(\beta c(t) - \alpha(t)h(t)\right)\mathrm{d}t \\
&\text{s.t.} \qquad \mathrm{d}F(t) = \left(r + \omega(t)\lambda_S\sigma_S\right)F(t)\mathrm{d}t + \omega(t)\sigma_S F(t)\mathrm{d}Z_S(t) + (Y(t)-c(t))\mathrm{d}t \\
&\qquad\qquad h(t) \le L(t) + c(t), \text{ for all } t \in [0,T].
\end{aligned}
\tag{2.8}
$$

The amount by which consumption $c(t)$ can fall below the reference level $h(t)$ is capped at $L(t)$, as follows from (2.8). The maximum possible loss $L(t)$ is assumed to be a deterministic function of time t.[19]

To solve (2.8), we define the stochastic discount factor $m(t)$. One can show that $m(t)$ satisfies the following dynamics (see, e.g., He and Pearson (1991)):

$$
\mathrm{d}m(t) = -rm(t)\mathrm{d}t + \phi^\top(t)m(t)\mathrm{d}Z(t), \tag{2.9}
$$

$m(0) > 0$. Here, $\top$ denotes the transpose sign, $Z(t) \equiv (Z_S(t), Z_Y(t))$, and $\phi(t)$ is a vector of factor loadings. We can determine $\phi(t)$ from the vector of prices of risk $\lambda(t) \equiv (\lambda_S, \lambda_Y(t))$ where $\lambda_Y(t)$ denotes the shadow price of labor income risk.[20] We have the following relationship:

$$
\phi(t) = -\begin{bmatrix} 1 & \rho_{SY} \\ \rho_{SY} & 1 \end{bmatrix}^{-1} \lambda(t). \tag{2.10}
$$

[19] This primarily technical condition ensures that consumption is bounded from below, thus avoiding the optimization problem to be ill-posed, and is required only if the individual is risk loving in the loss domain. Although in principle consumption can become negative, we can choose the maximum possible loss $L(t)$ such that the probability of consumption falling below zero is negligibly small.

[20] Since labor income risk is not traded and its market price cannot be observed, we refer to $\lambda_Y(t)$ as the *shadow* price of labor income risk (instead of the market price of labor income risk).

By the principle of no arbitrage, $\lambda_S = (\mu_S - r)/\sigma_S$. However, as labor income risk is non-tradable, the principle of no arbitrage does not uniquely determine the value of $\lambda_Y(t)$. Section 3 and the (online) appendix explain how we endogenously determine $\lambda_Y(t)$.

3 Optimal Life-Cycle Policies

The financial market as defined in Section 2.1 is incomplete, since the number of sources of risk (labor income and stock returns) exceeds the number of risky assets (the stock). We determine the optimal consumption and portfolio policies in an incomplete financial market as follows.

First, given the shadow price of labor income risk $\lambda_Y(t)$, we transform the dynamic optimization problem (2.8) into a static variational problem using the martingale approach.[21] We then convert this static primal problem into a static dual optimization problem without endogenous updating of the reference level using the isomorphism of Schroder and Skiadas (2002) (see also Van Bilsen et al. (2020b)).

Next, we endogenously determine the share of dual total wealth invested in the risky stock and the dual shadow price of labor income risk by imposing the condition that unexpected changes in dual tradable total wealth match unexpected changes in the value of future optimal dual consumption.[22] As a result, the dual shadow price of labor income risk is chosen such that the individual does not wish to hedge non-tradable uncertainty, i.e., the individual's demand for non-marketed consumption claims is zero. We then substitute this endogenously determined dual shadow price of labor income risk in the optimal dual consumption policy.

[21] See, e.g., Pliska (1986), Karatzas, Lehoczky, and Shreve (1987), and Cox and Huang (1989, 1991).

[22] Unexpected changes in dual tradable total wealth and in the value of future optimal dual consumption may arise from either a stock return shock or a labor income shock. Hence, the condition that unexpected changes in dual tradable total wealth match unexpected changes in the value of future optimal dual consumption leads to a system of two equations in two unknowns. We can now uniquely solve for the share of dual total wealth invested in the risky stock and the dual shadow price of labor income risk. Formally, this identifies the minimax local martingale measure (He and Pearson (1991)).

Finally, we use equivalence relationships to find the optimal (primal) consumption policy, the optimal share of (primal) total wealth invested in the risky stock, and the optimal (primal) shadow price of labor income risk. Note that although labor income risk is unhedgeable, optimal consumption is hedgeable. For details, we refer the reader to Appendix A, containing the static dual formulation, Appendix B, providing the derivation of the optimal policies and shadow price of risk, and the online appendix, describing the non-trivial numerical solution algorithm.

3.1 Optimal Consumption Choice

We are now ready to present the optimal consumption choice.

Theorem 1 (optimal consumption choice) *Consider an individual who solves the maximization problem* (2.8)*. Let* $\lambda_Y^*(t)$ *be the optimal (endogenous) shadow price of labor income risk. Denote by* $h^*(t)$ *the individual's optimal reference level at adult age* t *implied by substituting the optimal past consumption choices into* (2.6)*. Then:*

(i) If the individual is ***risk averse*** *in the loss domain, the optimal consumption choice at adult age* t *is given by*

$$c^*(t) = \begin{cases} h^*(t) + \left(\frac{ye^{\delta t}\widehat{m}(t)}{\gamma_G \widehat{m}(0)}\right)^{\frac{1}{\gamma_G - 1}}, & \text{if } \frac{\widehat{m}(t)}{\widehat{m}(0)} \leq \xi(t); \\ h^*(t) - \min\left\{\left(\frac{ye^{\delta t}\widehat{m}(t)}{\eta\gamma_L \widehat{m}(0)}\right)^{\frac{1}{\gamma_L - 1}}, L(t)\right\}, & \text{if } \frac{\widehat{m}(t)}{\widehat{m}(0)} > \xi(t); \end{cases} \tag{3.1}$$

where the threshold $\xi(t)$ is implicitly defined as follows:

$$
\begin{aligned}
0 = {} & e^{-\delta t}\left(1-\gamma_G\right)\left(\frac{y e^{\delta t}\xi(t)}{\gamma_G}\right)^{\frac{\gamma_G}{\gamma_G-1}} \\
& + e^{-\delta t}\eta \min\left\{\left(\frac{y e^{\delta t}\xi(t)}{\eta\gamma_L}\right)^{\frac{1}{\gamma_L-1}}, L(t)\right\}^{\gamma_L} \\
& - y\xi(t)\min\left\{\left(\frac{y e^{\delta t}\xi(t)}{\eta\gamma_L}\right)^{\frac{1}{\gamma_L-1}}, L(t)\right\}.
\end{aligned}
\tag{3.2}
$$

(ii) *If the individual is* ***risk loving*** *in the loss domain, the optimal consumption choice at adult age t is given by*

$$
c^*(t) = \begin{cases} h^*(t) + \left(\frac{y e^{\delta t}\widehat{m}(t)}{\gamma_G \widehat{m}(0)}\right)^{\frac{1}{\gamma_G-1}}, & \text{if } \frac{\widehat{m}(t)}{\widehat{m}(0)} \leq \zeta(t); \\ h^*(t) - L(t), & \text{if } \frac{\widehat{m}(t)}{\widehat{m}(0)} > \zeta(t); \end{cases}
\tag{3.3}
$$

where the threshold $\zeta(t)$ is implicitly defined as follows:

$$
0 = e^{-\delta t}\left(1-\gamma_G\right)\left(\frac{y e^{\delta t}\zeta(t)}{\gamma_G}\right)^{\frac{\gamma_G}{\gamma_G-1}} + e^{-\delta t}\eta L(t)^{\gamma_L} - y\zeta(t)L(t).
\tag{3.4}
$$

Here, $\widehat{m}^*(t) = m(t)\left(1+\beta f(t)\right)$ with

$$
f(t) \equiv \mathbb{E}_t\left[\int_t^{T_D} \frac{m(s)}{m(t)} e^{-\int_t^s (\alpha(u)-\beta)\mathrm{d}u}\mathrm{d}s\right]
\tag{3.5}
$$

is the dual stochastic discount factor associated with the optimal shadow price of labor income risk $\lambda_Y^*(t)$; and the Lagrange multiplier y is chosen such that the static budget constraint, as defined in (A8) in Appendix A, holds with equality.

Proof. See Appendix B.1.

In the online appendix, we describe how to numerically determine the optimal shadow price of labor income risk $\lambda_Y^*(t)$. This reveals that the optimal shadow price of labor income risk is, under some mild conditions, proportional to the ratio of dual human wealth to dual total wealth (i.e., discounted value of dual future consumption $\widehat{c}(s)$, with $s > t$). This result is consistent with Sangvinatsos and Wachter (2005) and de Jong (2008), who find that under standard CRRA preferences, the shadow price of labor income risk is proportional to the ratio of human wealth to total wealth.

3.2 Optimal Portfolio Choice

The following theorem summarizes the optimal portfolio choice.

Theorem 2 (optimal portfolio choice) *Consider an individual who solves the maximization problem* (2.8). *Let*

$$\phi^*(t) = -\begin{bmatrix} 1 & \rho_{SY} \\ \rho_{SY} & 1 \end{bmatrix}^{-1} \times \begin{bmatrix} \lambda_S \\ \lambda_Y^*(t) \end{bmatrix} \tag{3.6}$$

be the vector of optimal factor loadings, and let $F(t)$ be the individual's financial wealth at adult age t. Then the optimal portfolio choice at adult age t is given by

$$\omega^*(t) = (1 + \beta f(t)) \frac{\partial \widehat{F}(t)}{\partial \widehat{m}(t)} \frac{\phi_S^*(t)}{\sigma_S} \frac{\widehat{m}^*(t)}{F(t)}, \tag{3.7}$$

where

$$\widehat{F}(t) = \frac{F(t) - f(t)h(t)}{1 + \beta f(t)} \tag{3.8}$$

and $\widehat{m}^*(t)$ is the dual stochastic discount factor associated with the optimal shadow price of labor income risk $\lambda_Y^*(t)$.

Proof. See Appendix B.2.

4 Main Implications

In this section, we explore our main implications.

4.1 Benchmark Setting

We consider an individual who starts working at age 25, retires at age 65, and passes away at age 85. Each period, the individual decides what share of labor income to consume and save, and how much to invest in the riskless asset and the risky stock. We assume the following parameter values for the financial market and the labor income process. The riskless interest rate r and the mean stock return μ_S are set at 1% and 5%, respectively. We assume that the stock return volatility σ_S is equal to 20%. These parameter values roughly coincide with those in Gomes, Kotlikoff, and Viceira (2008). Initial salary $Y(0)$ is equal to \$68,000 per year, which corresponds to the average US household income in 2019. During retirement, the individual does not receive labor income. In line with Viceira (2001), we set labor income volatility σ_Y equal to 10%.[23] Moreover, the growth rate of median labor income is constant and normalized to zero, i.e., $\mu_Y(t) = 0$ for every age, and labor income shocks are permanent, i.e., $\kappa = 0$. In Section 4.4, we explore the impact of different values of κ on the optimal policies. The initial reference level equals 50% of initial labor income. We set the parameters that characterize the reference level, i.e., $\alpha(t) = \alpha$ and β, equal to 0.2 and 0.1, respectively. Furthermore, we assume that the subjective rate of time preference δ equals 4%, the loss aversion index η is equal to 2, γ_G equals 0.4, and γ_L is equal to 1.3, in line with Van Bilsen et al. (2020b). Finally, initial financial wealth $F(0)$ is zero. Our main implications remain qualitatively unchanged if we vary the values of the parameters within reasonable limits and if we consider a different process for labor income. Finally, we note that the individual's relative risk aversion is time-varying. In most scenarios, relative risk aversion is in between 1 and 3.

[23] Viceira's estimate of σ_Y is based on Chamberlain and Hirano (1999) and Caroll and Samwick (1997).

4.2 Optimal Consumption Choice

This subsection explores the individual's optimal consumption choice.

4.2.1 Excess Sensitivity of the Current Optimal Savings Rate

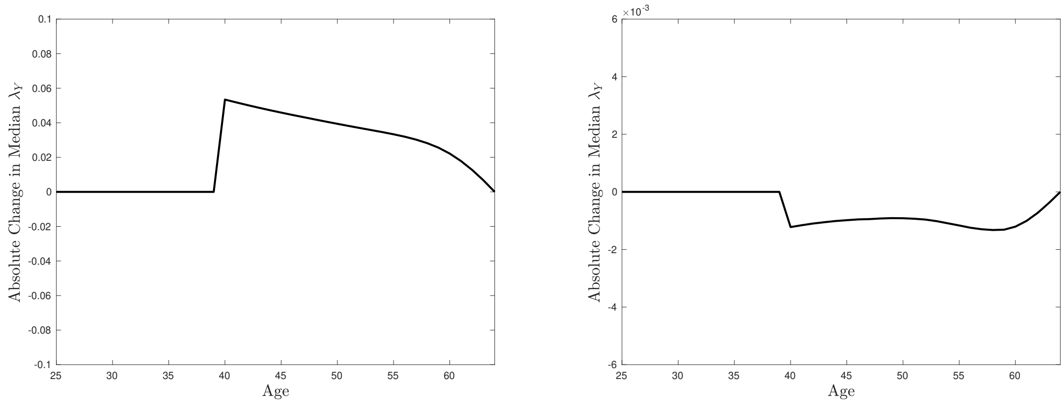
We aim to analyze the implications of a stock return shock and a labor income shock for optimal consumption and the optimal savings rate. Before doing so, we examine how total wealth responds to a shock. Indeed, total wealth finances optimal consumption. The effect of a shock on total wealth is not immediately clear, as shocks to both stock prices and individual labor income affect total wealth indirectly through their impact on the shadow price of labor income risk.

Obviously, a negative stock return shock reduces financial wealth. Furthermore, we find that a drop in the stock price leads to a decrease in human wealth. Indeed, a lower stock price implies a higher shadow price of labor income risk, which, in turn, implies a lower expected discounted value of future labor income; see Figure 1(a).[24] Hence, a stock price decline constitutes additional bad news, as both financial wealth and human wealth decrease. We note that, even in the absence of correlation between stock return shocks and labor income shocks, human wealth decreases following a stock price decline.

While a drop in labor income leaves current financial wealth unaffected, it has two counteracting effects on human wealth. On the one hand, human wealth decreases, because expected labor income declines, assuming the labor income shock is non-transitory. On the other hand, human wealth increases, because the shadow price of labor income risk becomes smaller; see Figure 1(b). We find that, for a wide range of parameter values, the first effect (heavily) dominates the second effect. Hence, human wealth decreases following a negative shock in labor income, which is economically

[24] Since the shadow price of labor income risk is – under some conditions – proportional to the ratio of dual human wealth to dual total wealth, a reduction in dual financial wealth leads to a higher shadow price of labor income risk. We refer the reader to (IA98) in the internet appendix for more information.

intuitive.



(a) Impact of Stock Return Decline (b) Impact of Labor Income Decline

Figure 1. Impact of Shocks on Median Shadow Price of Labor Income Risk The figure shows how the median shadow price of labor income risk $\lambda_Y(t)$ changes following a 30% drop in the stock price at age 40 (left panel) and following a 20% permanent drop in labor income at age 40 (right panel). The parameter values are given in Section 4.1.

From our analysis above, it becomes clear that both a negative stock return shock and a negative permanent labor income shock lead to a reduction in the individual's total wealth. As a result, the individual must reduce future median optimal consumption levels following a negative shock. As illustrated in Figures 2 and 3, we find that optimal consumption responds sluggishly to both shocks. Intuitively, with an endogenous reference level, consumption reductions in the far future are felt less acutely than consumption reductions in the near future. The dashed lines show the case in which the individual has conventional CRRA preferences. In that case, a shock has a similar effect on median consumption levels in the near future as in the distant future, implying no gradual adjustment.

A direct implication of the reduction of median optimal consumption following negative shocks is that the current optimal savings rate – i.e., the optimal share of labor income saved – responds differently to a stock return shock than to a labor income shock; see Figure 4(a). A stock price decline – which leads to a reduction in financial

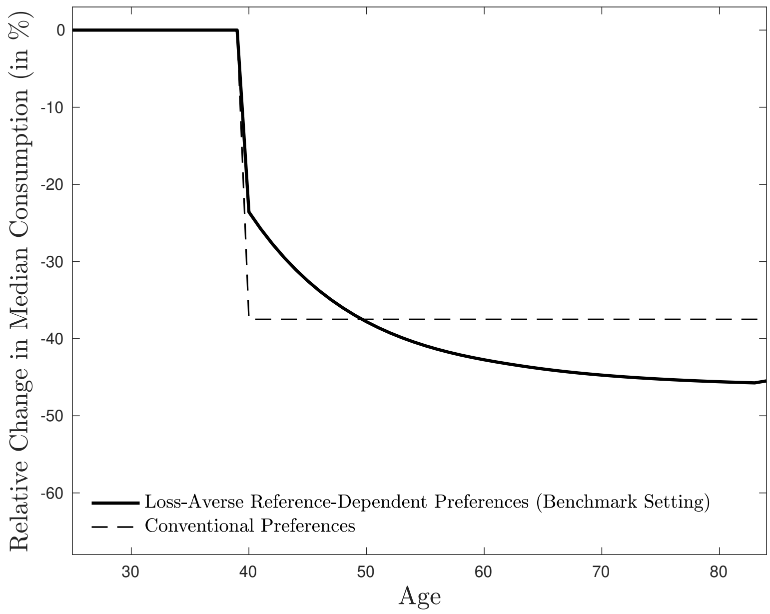


Figure 2. Impact of a Stock Return Decline on Future Median Optimal Consumption The figure shows the impact of a 30% drop in the stock price at age 40 on future median optimal consumption. The solid line illustrates an individual with loss-averse reference-dependent preferences with parameter values given in Section 4.1 (benchmark setting). The dashed line illustrates the case in which the individual has conventional CRRA preferences (with unit relative risk aversion) and does not face labor income risk. Both cases experience the same relative decline in total wealth.

wealth – affects consumption not only during the retirement phase but also during the working phase. Under conventional CRRA preferences, she increases the optimal savings rate after a drop in the current stock price, to compensate for the reduction in financial wealth. However, in the case of loss aversion and an endogenous reference level, the increase in the optimal savings rate is lower compared to conventional preferences. Indeed, with loss aversion and an endogenous reference level, the individual prefers to protect current consumption. The solid line in Figure 4(a) shows how the current optimal median savings rate responds to a stock return shock.

The dash-dotted line in Figure 4(a) displays that, after a permanent drop in current labor income, the current optimal median savings rate decreases. We can decompose the optimal response of the current savings rate into two parts. The first part reflects

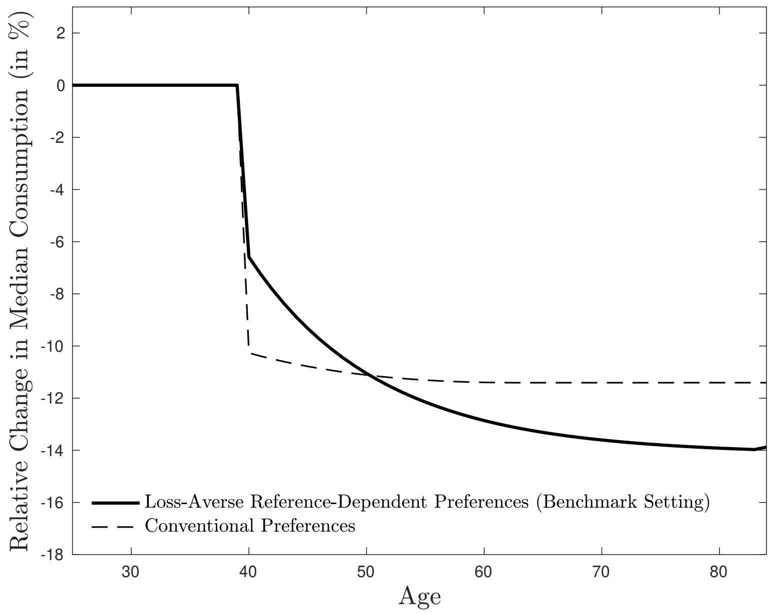
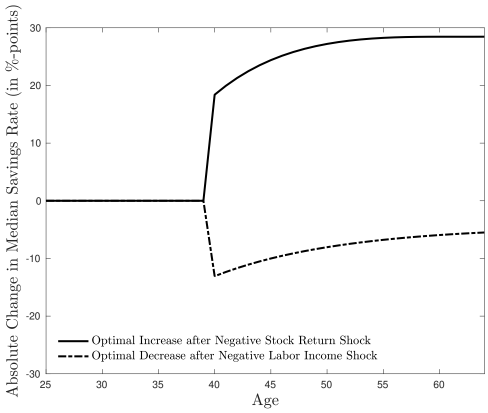
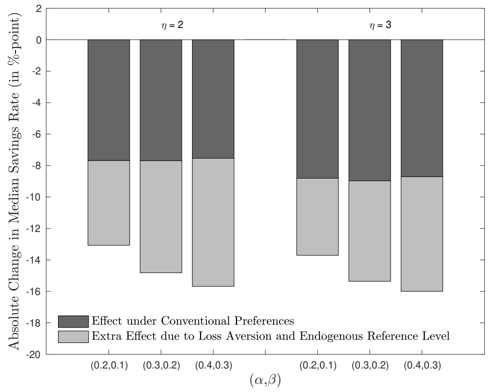


Figure 3. Impact of a Labor Income Decline on Future Median Optimal Consumption The figure shows the impact of a 20% permanent drop in labor income at age 40 on future median optimal consumption. The solid line illustrates an individual with loss-averse reference-dependent preferences with parameter values given in Section 4.1 (benchmark setting). The dashed line illustrates the case in which the individual has conventional CRRA preferences (with relative risk aversion equal to 5) and faces labor income risk. Both cases experience the same relative decline in total wealth.

the preference for consumption smoothing, which is also present under conventional preferences, as illustrated by the dark gray area in Figure 4(b). Due to consumption smoothing, a labor income shock affects consumption not only during the working phase, but also during the retirement phase. To shift part of the current labor income decline to the retirement period, the individual reduces her optimal savings rate, leaving less financial wealth available for retirement consumption. The second part is an additional effect arising from loss aversion and the endogenous reference level, as illustrated by the light gray area in Figure 4(b). Indeed, since the individual prefers to protect current consumption, she reduces her optimal savings rate even further. As a result, the effect on retirement consumption of a current labor income shock will be larger under loss-averse reference-dependent preferences than under conventional preferences. Figure 4(b) demonstrates that this result is robust across different parameter combinations. We conclude that the current optimal savings rate is excessively sensitive, i.e., it over-responds, to a permanent labor income shock.



(a) Absolute Change in Savings Rate



(b) Decomposition of Optimal Response

Figure 4. Impact of Declines on Optimal Savings Rate The left panel of this figure shows how the median optimal savings rate responds following a 30% drop in the stock price at age 40 and following a 20% permanent drop in labor income at age 40. The right panel decomposes, for various values of α, β and η, the optimal response of the current optimal savings rate to a 20% permanent drop in labor income at age 40. The first part of this decomposition is already present under conventional CRRA preferences (dark gray area), while the second part is an extra effect and is due to loss aversion and an endogenous reference level (light gray area). The remaining parameter values are given in Section 4.1.

Figure 5 shows the post-shock savings rates for a wide range of income shocks. We assume that the ratio of consumption to the reference level and the ratio of financial to human wealth equal 2 and 0.5, respectively. Also, the original, pre-shock, savings rate equals 22%. We find that, for a wide range of shocks, i.e., scenarios in which income declines exceed 40%, the individual does not save at all for a period and instead withdraws financial wealth prior to retirement, as shown by the gray area in Figure 5. This result is even stronger for lower original, pre-shock, savings rates. As shown by Figure 4(b), the preference for dissaving is substantially larger under loss-averse reference-dependent utility than under conventional utility. Hence, an institutional setting that restricts access to financial wealth before retirement, such as a 401(k) plan or Individual Retirement

Account (IRA), can induce substantial welfare losses; see also Section 4.5.

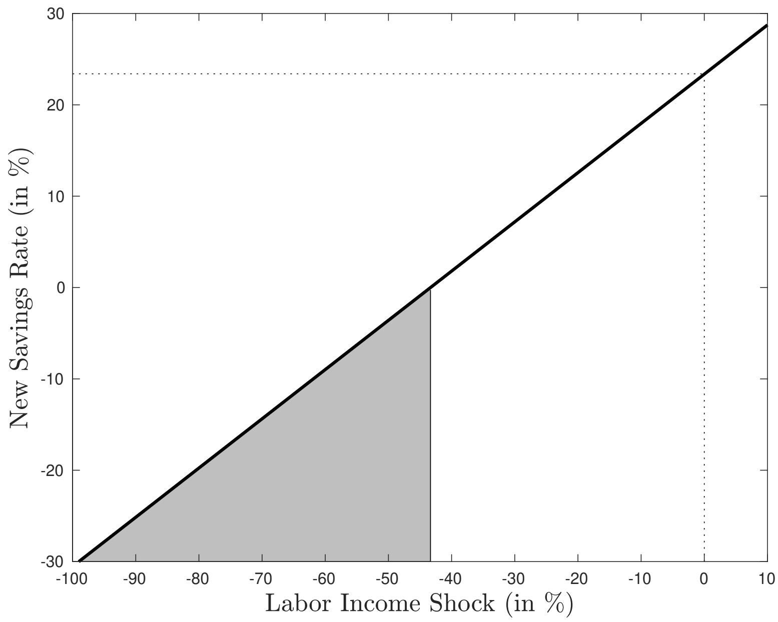


Figure 5. Post-shock Savings Rate for a Wide Range of Labor Income Shocks The figure shows the new (post-shock) optimal savings rate for a wide range of labor income shocks. We assume that the original, pre-shock, optimal savings rate is equal to 22%. Furthermore, the ratio of consumption to the reference level and the ratio of financial wealth to human wealth are assumed to be equal to 2 and 0.5, respectively. The gray area indicates the scenarios in which the optimal savings is negative, i.e., the individual withdraws financial wealth prior to retirement. The parameter values are given in Section 4.1.

4.2.2 Heterogeneity in Optimal Response of Savings Rate

The optimal response of the current savings rate varies (substantially) with both the ratio of consumption to the reference level and the ratio of financial wealth to human wealth. Figure 6 illustrates the optimal response as a function of the ratio of consumption to the reference level. Each dot in this figure corresponds to a different ratio of financial wealth to human wealth. We find that when current consumption is close to the reference level, the optimal savings rate typically falls sharply following a permanent drop in labor income; see Figure 6. Intuitively, as consumption approaches the reference level, the individual strongly prefers to protect current consumption.

Accordingly, after a permanent labor income shock, she reduces the optimal savings rate substantially. Finally, under conventional preferences, the response of the optimal savings rate is independent of the gap between the individual's consumption level and reference level, as this gap does not enter a conventional preference specification.

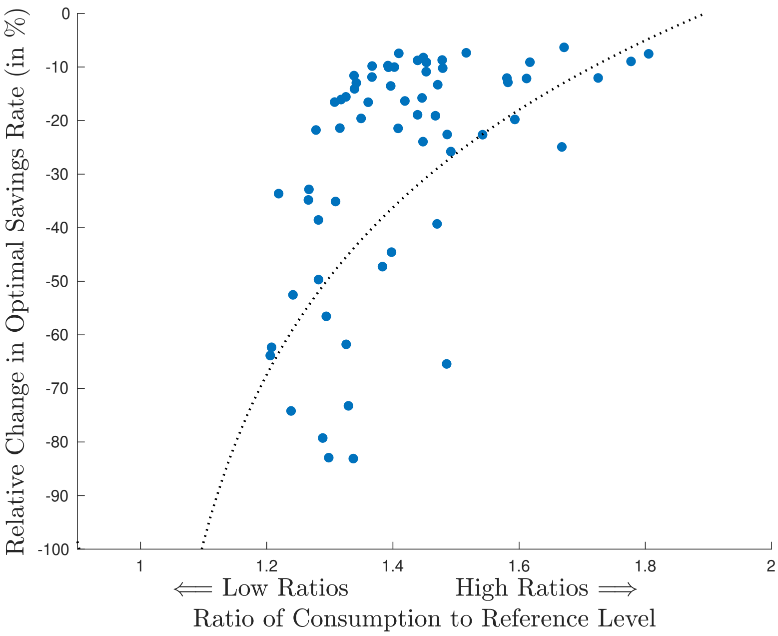


Figure 6. Heterogeneity in Optimal Savings Rate The figure shows the optimal response of the current savings rate to a 5% permanent drop in labor income at age 30 as a function of the ratio of consumption to the reference level. Each dot in this figure corresponds to a different ratio of financial wealth to human wealth. The initial reference level is chosen to be equal to 80% of initial labor income and $\alpha = 15\%$, to illustrate the strong preference for dissaving at low ratios of consumption to the reference level. The remaining parameter values are given in Section 4.1.

Furthermore, we find that when financial wealth is large relative to human wealth, the individual adjusts the optimal savings rate considerably. Intuitively, if a worker's financial wealth dominates her human wealth, a permanent drop in current labor income affects total wealth only marginally, so there is little need to adjust consumption heavily. Since total wealth finances consumption, a worker with substantial financial wealth relative to human wealth absorbs the labor income shock primarily by reducing her optimal savings rate.

4.3 Optimal Portfolio Choice

Next, we explore the individual's optimal portfolio choice.

4.3.1 Excess Sensitivity of the Current Optimal Portfolio Share

This subsection analyzes the implications of both a stock return shock and a permanent labor income shock for the optimal share of financial wealth invested in the risky stock, i.e., the portfolio share. In case of risky labor income and loss-averse reference-dependent preferences, the impact of a stock return shock and a labor income shock on the optimal portfolio share is ambiguous. It is clear that a negative stock return shock leads to lower financial wealth. However, human wealth also decreases following a drop in the stock price, as the shadow price of labor income risk goes up; see Figure 1(a). Because both financial and human wealth decrease, it is not immediately obvious how the individual should adjust her optimal share of financial wealth invested in the stock.

Our results show that the impact of a stock return shock on financial wealth is typically much larger than on human wealth, which is economically intuitive. Hence, under conventional CRRA preferences, she increases the optimal share of financial wealth invested in the stock following a decline in the stock price, to maintain the same exposure of total wealth to stock return risk as before the shock; see the dashed line in Figure 7(a). Under loss-averse reference-dependent preferences, however, a negative stock return shock typically leads to an increase in the individual's relative risk aversion. In this case, a relatively large share of total wealth must be allocated to the riskless asset, to ensure that future consumption levels exceed future reference levels with high probability. As a result, the increase of the optimal risky portfolio share following a stock price decline is somewhat less pronounced under loss-averse reference-dependent preferences than under conventional preferences; see Figure 7(a).

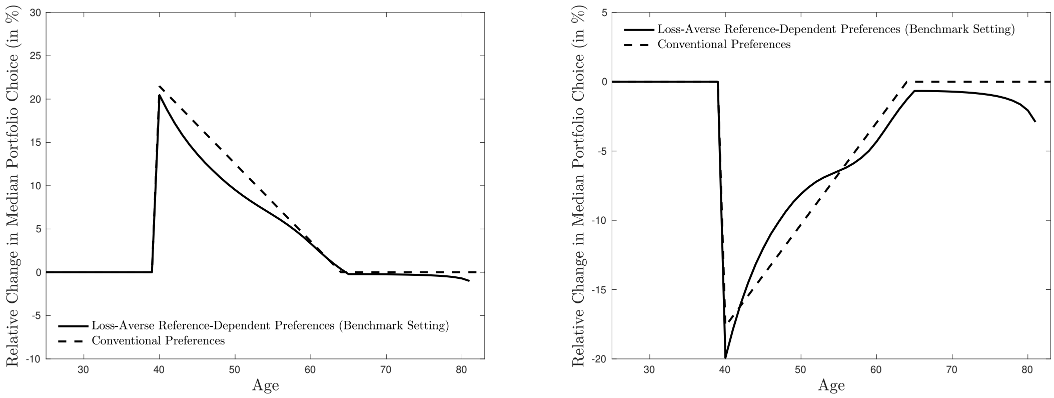
As discussed in Section 4.2.1, a negative permanent labor income shock has two counteracting effects on human wealth. The net effect is a decrease in human wealth following a permanent decline in labor income. As a result, under conventional CRRA preferences, the individual reduces her optimal share of financial wealth invested in the stock following a negative labor income shock, in order to maintain the same exposure of total wealth to stock return risk as before the shock; see the dashed line in Figure 7(b). Under loss-averse reference-dependent preferences, relative risk aversion typically increases following a negative permanent labor income shock. Indeed, the individual's current consumption moves closer to her reference level, assuming current consumption exceeds the current reference level. This provides an additional reason, not present under conventional preferences, for the individual to immediately reduce her optimal risky portfolio share; see Figure 7(b).

Figure 8 illustrates this additional downward adjustment of the optimal portfolio share (in percentage points) relative to the optimal response under conventional preferences for different parameter values, highlighting the robustness of our finding. We conclude that both the optimal savings rate and the optimal portfolio share are excessively sensitive, i.e., they over-respond, to labor income shocks.

4.3.2 Heterogeneity in Optimal Response of Portfolio Share

Not only does the optimal savings rate respond to the ratio of consumption to the reference level, but the optimal portfolio share does as well (see Figure 9). A key difference between Figure 6 – which illustrates heterogeneity in the optimal savings rate – and Figure 9 is that the optimal portfolio share is almost never negative in absolute terms, whereas this is not the case for the optimal savings rate.

The shape of Figure 9 shows how the optimal response of the portfolio share varies with the ratio of consumption to the reference level. We find that when this ratio is high (low), the optimal portfolio share is relatively insensitive (sensitive) to a permanent drop



(a) Impact of Stock Return Decline

(b) Impact of Labor Income Decline

Figure 7. Impact of Shocks on Optimal Portfolio Share The figure shows how the optimal share of financial wealth invested in the risky stock changes as a result of a shock at age 40. The left panel shows the optimal response to a 5% decline in the stock price while the right panel illustrates the optimal response to a 20% permanent decline in labor income. The solid lines illustrate an individual with loss-averse reference-dependent preferences with parameter values given in Section 4.1 (benchmark setting). The dashed lines illustrate the case in which the individual has conventional CRRA preferences and faces labor income risk. The parameter values are given in Section 4.1.

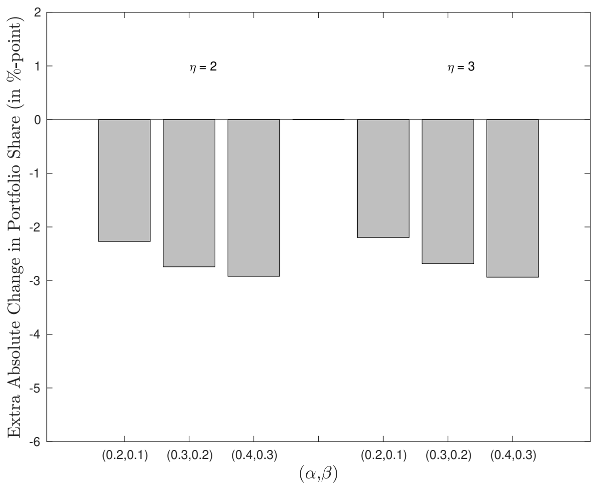


Figure 8. Impact of Income Decline on Optimal Portfolio Share The figure illustrates the additional downward adjustment of the optimal portfolio share (in percentage points) following a 20% permanent decline in labor income, relative to the optimal response under conventional preferences, across different parameter values. The remaining parameter values are given in Section 4.1.

in labor income. Intuitively, when the individual's consumption is high relative to the reference level, a permanent labor income shock affects her relative risk aversion only slightly, reducing the need to de-risk. Under CRRA utility, this behavior does not occur, as relative risk aversion remains constant in all circumstances.

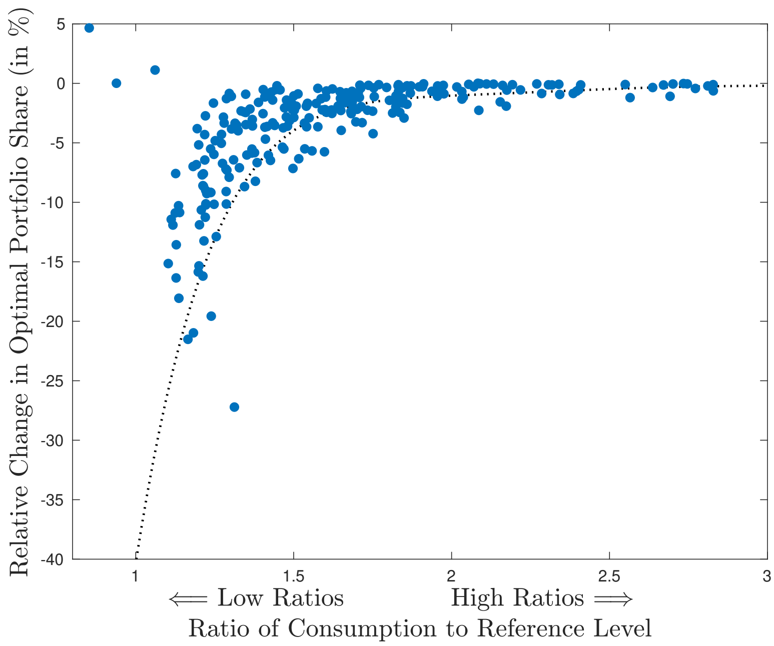


Figure 9. Heterogeneity in Optimal Portfolio Share The figure shows the optimal response of the current portfolio share to a 20% permanent drop in labor income at age 60 as a function of the ratio of consumption to the reference level. The initial reference level is chosen to be equal to 80% of initial labor income and $\alpha = 15\%$, to illustrate the strong preference for de-risking at low ratios of consumption to the reference level. The remaining parameter values are given in Section 4.1.

We also find that the optimal response of the portfolio share varies with the ratio of human wealth to financial wealth. In particular, when human wealth is small relative to financial wealth, the individual adjusts her optimal portfolio share only minimally. Intuitively, if a worker's human wealth is small compared to her financial wealth, a drop in current labor income affects total wealth only slightly, so there is little reason to substantially adjust her portfolio share.

4.3.3 More Conservative at Old Ages and Less Conservative at Young Ages

In a setting with labor income risk and loss-averse reference-dependent preferences, the individual still follows a life-cycle portfolio strategy. That is, the optimal share of financial wealth invested in the stock generally decreases with age. This is consistent with conventional wisdom (see, e.g., Bodie et al. (1992)). However, for our parameter values, the optimal portfolio strategy is typically more conservative at older ages and less conservative at younger ages compared to the case with riskless labor income and conventional CRRA preferences; see Figure 10. This pattern holds even in the absence of correlation between stock return shocks and labor income shocks.[25]

Both risky labor income and loss-averse reference-dependent preferences affect the optimal portfolio strategy. Figure 11(a) illustrates the impact of risky labor income on the optimal portfolio strategy. We find that risky labor income causes the optimal portfolio share to decrease, especially at younger ages. Indeed, due to a positive shadow price of labor income risk, human wealth, which is equal to the discounted value of future labor income, becomes smaller. The individual thus holds less human wealth, compared to the case with no labor income risk. As a result, the individual should invest less in risky stocks to achieve the optimal exposure of total wealth to stock return risk. We find that in our setting, the median portfolio strategy at age 40 decreases by 25 percent compared to the case with conventional CRRA preferences – with relative risk aversion equal to 5 – and no labor income risk.

Reference-dependence introduces two additional counteracting effects on the optimal portfolio choice, as illustrated in Figure 11(b). The first effect arises from the reference level itself. Under a "strict" reference level, this effect implies that the individual must

[25] The impact of the correlation coefficient on the optimal portfolio strategy is twofold. First, a higher ρ_{SY} means that human wealth carries more stock return risk, so that the individual's willingness to invest in the risky stock decreases. Second, a higher ρ_{SY} leads, in our setting, to a higher shadow price of labor income risk, which causes human wealth to decrease. This, in turn, leads to a reduction of the optimal portfolio share.

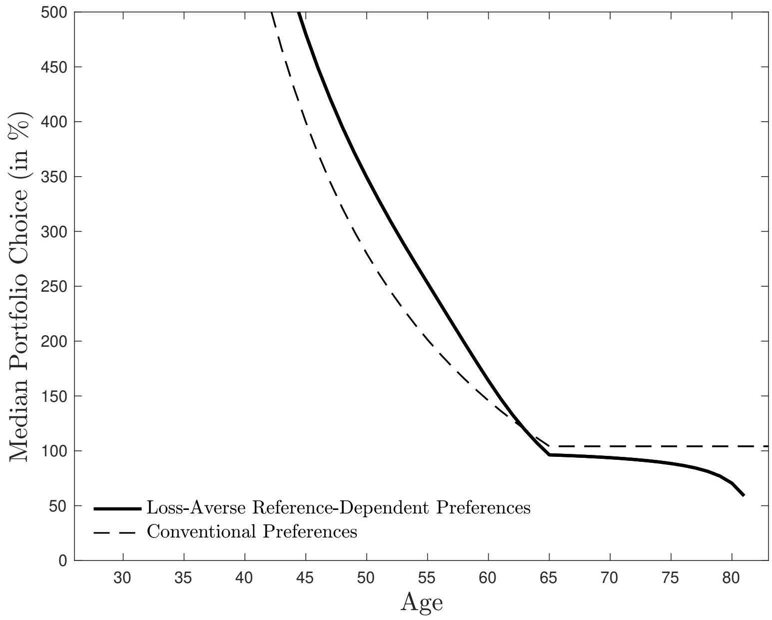
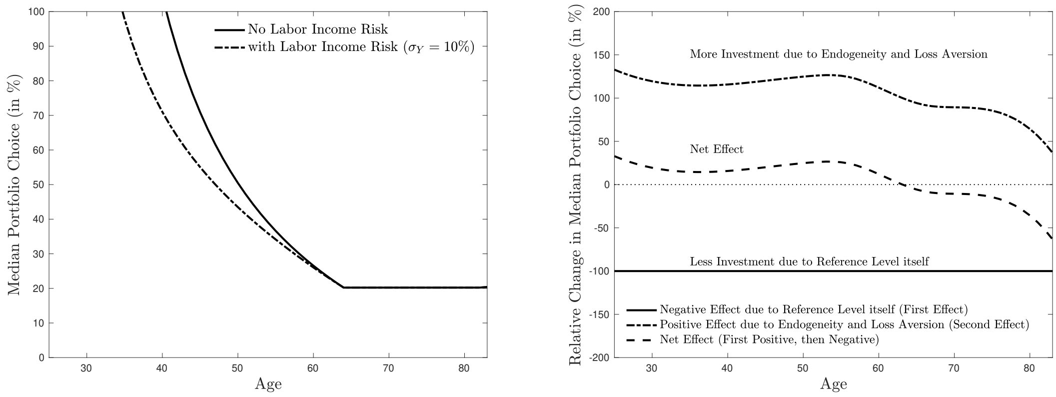


Figure 10. Optimal Portfolio Share The figure shows the median optimal share of financial wealth invested in the risky stock as a function of age. The solid line illustrates an individual with loss-averse reference-dependent preferences. The dashed line illustrates the case in which the individual has conventional CRRA preferences (with unit relative risk aversion) and does not face labor income risk. The initial reference level is equal to 80% of initial labor income and $\alpha = 15\%$, to strengthen the impact of the reference level on the optimal portfolio share. The remaining parameter values are given in Section 4.1

reserve part of her financial wealth to guarantee that future consumption levels exceed future reference levels with certainty. As a result, the optimal portfolio share decreases. This first effect is illustrated by the solid line in Figure 11(b), which shows the decrease in the median portfolio strategy compared to standard CRRA preferences with relative risk aversion equal to one. The second effect stems from the endogeneity of the reference level and loss aversion. With an endogenous reference level, the individual will gradually adjust consumption to shocks, thereby limiting the impact of a shock on current consumption. This allows the individual to take more investment risk. Additionally, loss aversion permits greater investment risk, since consumption no longer needs to exceed the strict reference level with certainty. The second effect is illustrated by the dash-dotted line in Figure 11(b), which shows the increase in the median portfolio strategy compared to preferences with a guarantee, i.e., with a strict reference level, assuming the guarantee equals 80% of initial labor income. We observe that, for our parameter setting, the first effect dominates the second effect at older ages and the second effect dominates the first effect at younger ages; see the dashed line in Figure 11(b).



(a) Impact of Non-Tradable Labor Income

(b) Impact of Reference-Dependence

Figure 11. Impact of Risky Labor Income and Loss-Averse Reference-Dependent Preferences The left panel shows the impact of risky labor income on the optimal portfolio share, while the right panel illustrates the impact of loss-averse reference-dependent preferences on the optimal share. The solid line in the right panel shows the decrease in the median portfolio strategy compared to conventional CRRA preferences (with relative risk aversion equal to one), while the dash-dotted line in the right panel shows the increase in the median portfolio strategy compared to preferences with a guarantee (assuming the guarantee equals 80% of initial labor income). The initial reference level is chosen to be equal to 80% of initial labor income and $\alpha = 15\%$, to strengthen the impact of the reference level on the optimal portfolio share. The remaining parameter values are given in Section 4.1.

4.4 Robustness Checks for Type of Labor Income Shock

So far, we have assumed that labor income shocks are permanent, i.e., $\kappa = 0$. We now conduct robustness checks to examine how our results vary with different types of labor income shocks, including permanent ($\kappa = 0$) and non-permanent shocks ($\kappa > 0$). These checks help verify the consistency of our findings regarding optimal consumption, savings,

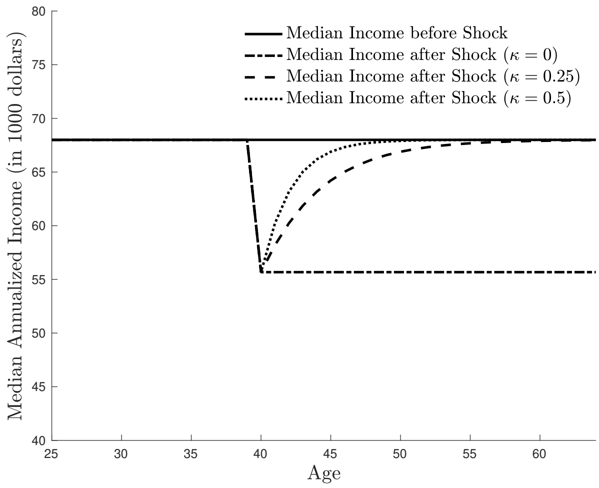


Figure 12. Impact of Different Types of Labor Income Declines on Future Median Consumption The figure shows the impact of a 20% drop in labor income at age 40 on future median income for various values of κ. The remaining parameter values are given in Section 4.1.

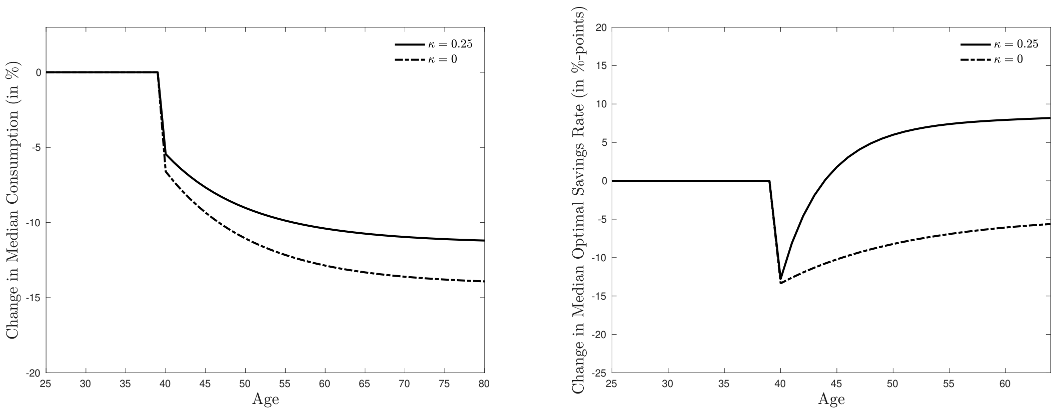
and portfolio choices.[26] More specifically, we consider the case where $\kappa = 0.25$. Figure 12 shows how median income behaves following a 20% decline in labor income at age 40 for different types of labor income shocks. In case $\kappa = 0.25$, we observe that at age 60, the individual's labor income is roughly equal to labor income before the shock.

4.4.1 Optimal Consumption Choice

Figure 13(a) illustrates the relative change in median consumption as a result of a 20% decline in labor income at age 40. If we compare the case $\kappa = 0.25$ with the case $\kappa = 0$ (permanent labor income shock), we observe, as expected, that the relative change in median consumption is smaller for the less permanent shock. Indeed, when labor income shocks are less persistent, their impact on the individual's total wealth is smaller. Furthermore, in both cases, the individual gradually absorbs the labor income shock:

[26]We can easily extend our numerical method to the case $\kappa > 0$. While the value of κ affects how labor income evolves over time, it essentially does not affect the numerical determination of the optimal policies and the shadow price of labor income risk. Indeed, the optimal strategy is a function of the current state variables, (dual) labor income and the (dual) stochastic discount factor.

with an endogenous reference level, it is optimal to postpone reductions in consumption.



(a) Optimal Consumption

(b) Optimal Savings Rate

Figure 13. Impact of Declines on Optimal Consumption and Savings Rate The left panel of this figure shows the impact of a 20% drop in labor income at age 40 on future median optimal consumption, while the right panel shows how the median optimal savings rate responds following a 20% drop in labor income at age 40. We consider $\kappa = 0.25$ and $\kappa = 0$ (permanent labor income shock). The remaining parameter values are given in Section 4.1.

Next, we analyze how the optimal savings rate responds to labor income shocks. Figure 13(b) illustrates the optimal response of the median savings rate following a 20% drop in labor income at age 40. After the labor income shock, the individual immediately decreases her optimal savings rate, to protect current consumption. She also seeks to transfer part of the labor income shock to the retirement phase. We also observe this behavior when $\kappa = 0$ (permanent labor income shock). What is new is that the optimal savings rate now rapidly converges back to the original, pre-shock, savings rate. After some time, the individual even saves more. This pattern arises because the shock is gradually absorbed and temporary: income eventually returns to its previous level while optimal consumption remains relatively low. We conclude that income shocks (permanent or non-permanent) have a substantial impact on the *current* optimal savings rate.

4.4.2 Optimal Portfolio Choice

This subsection explores the impact of permanent and non-permanent labor income shocks on the optimal portfolio share. Figure 14 illustrates the optimal response of the median portfolio share following a 20% drop in labor income at age 40. Comparing the case where $\kappa = 0.25$ to that with $\kappa = 0$ (permanent labor income shock), we first observe that the instantaneous drop in the optimal portfolio share is smaller in the case where $\kappa = 0.25$. Indeed, when labor income shocks are less permanent, their effect on total wealth is smaller, reducing the individual's preference to de-risk. Furthermore, in the median scenario for both cases, the optimal portfolio share eventually increases after several years, and this rebound is substantially stronger when $\kappa = 0.25$. As income returns to its original, pre-shock level while optimal consumption remains relatively low, the individual can afford to take on more investment risk.

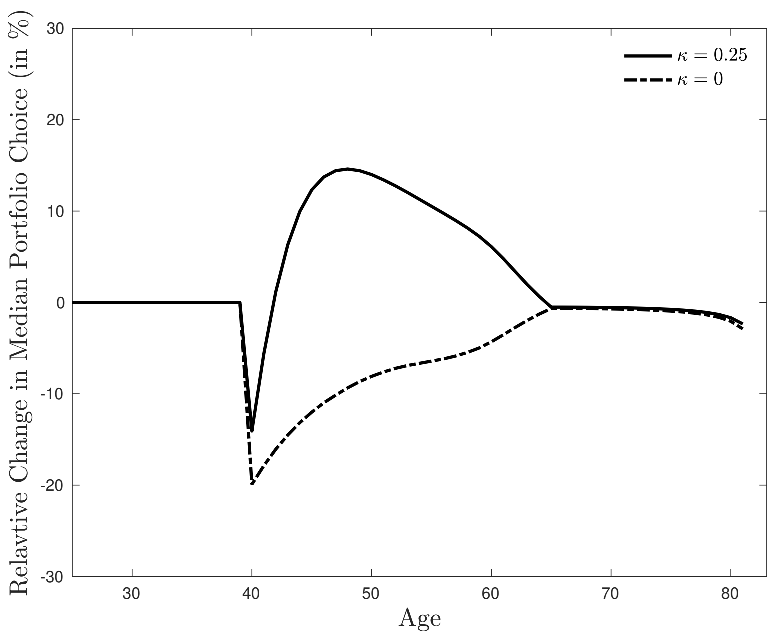


Figure 14. Impact of Labor Income Decline on Optimal Portfolio Share The figure shows how the optimal share of financial wealth invested in the risky stock changes as a result of a 20% decline in labor income at age 40. We consider $\kappa = 0.25$ and $\kappa = 0$ (permanent labor income shock). The remaining parameter values are given in Section 4.1.

4.5 Welfare Implications of Suboptimal Strategies

This subsection evaluates the welfare costs arising from incorrectly assuming conventional CRRA preferences. Our setup is as follows. An individual with loss-averse reference-dependent preferences facing labor income risk delegates her consumption and portfolio decisions to an asset manager. However, this asset manager does not implement the individual's true optimal strategies and instead assumes she has CRRA preferences.

The individual is free to choose the value of the relative risk aversion coefficient at the beginning of her life-cycle, which determines the shape of the CRRA utility function. We assume that the individual is smart and selects the coefficient to minimize the difference between her true optimal utility – associated with the actual optimal policies – and the sub-optimal utility – associated with the conventional CRRA-based policies.

Table 1 reports our results. We measure welfare losses in terms of the relative decline in certainty equivalent consumption. As shown in Table 1, strategies in which the savings rate responds limitedly to a labor income shock — as occurs under conventional CRRA preferences — entail substantial welfare costs for our loss-averse individual. For our benchmark setting (i.e., $\alpha(t) = \alpha = 0.2$, $\beta = 0.1$ and $\eta = 2$), we find a minimum lifetime welfare loss of 26.86%. Thus, while normatively relevant, CRRA policies are detrimental to the welfare of our individual. Importantly, the welfare losses remain substantial across alternative parameter values.

We also compute welfare losses associated with a common, state-independent consumption and portfolio strategy. Specifically, in each period the loss-averse individual withdraws $1/T$ of her total wealth, where T denotes remaining lifetime and invests "100 minus her age" percent of her financial wealth in risky assets. The results are reported in Table 2 for different values of $\alpha(t) = \alpha$, β and η. We find that welfare losses are substantially larger than those reported in Table 1, which presents the welfare losses associated with the implementation of the Merton strategy. This indicates that

Table 1. **Welfare Costs** The table reports the minimum lifetime welfare losses associated with inadequately assuming CRRA preferences in which the savings rate does not respond excessively sensitive to a labor income shock. We compute the minimum lifetime welfare loss for different values of $\alpha(t) = \alpha$, β and η. We measure welfare losses in terms of the relative decline in certainty equivalent consumption. The remaining parameter values are given in Section 4.1.

true parameters α, β and η			
α	β	η	minimum welfare loss (in %)
0.2	0.1	2	26.86
0.3	0.2	2	47.03
0.4	0.3	2	69.59
0.2	0.1	3	29.55
0.3	0.2	3	47.73
0.4	0.3	3	69.87

the use of naïve, state-independent strategies can lead to considerably larger welfare losses.

Table 2. **Welfare Costs** The table reports the lifetime welfare losses associated with a common, state-independent consumption and portfolio strategy. We compute the lifetime welfare loss for different optimal values of $\alpha(t) = \alpha$, β and η. We measure welfare losses in terms of the relative decline in certainty equivalent consumption. The remaining parameter values are given in Section 4.1.

true parameters α, β and η			
α	β	η	welfare loss (in %)
0.2	0.1	2	62.21
0.3	0.2	2	66.54
0.4	0.3	2	80.28
0.2	0.1	3	67.09
0.3	0.2	3	71.39
0.4	0.3	3	82.00

4.6 Testable Implications

We can compare the main implications of our model featuring loss-averse reference-dependent preferences and risky labor income with data on savings behavior and portfolio holdings, to provide support for our main implications. In this subsection, we briefly describe how the main implications can be tested.

Our first main implication is that both optimal savings and portfolio choices are more sensitive to labor income shocks than what is predicted by standard models with CRRA preferences. This excess sensitivity leads to excessively smooth consumption patterns – meaning that individuals adjust their consumption less in response to income changes than traditional theory would suggest; see Figure 3. In essence, consumption tends to under-respond to income shocks. This prediction can be tested empirically by examining how changes in income relate to changes in consumption over time. If consumption responses are consistently smaller than what CRRA-based models predict, this provides support for the model's implication. A similar approach can be applied to portfolio choices: by studying how portfolio shares respond to income fluctuations, one can assess whether the observed sensitivity exceeds that predicted by standard models with CRRA preferences. Several empirical studies have found clear evidence of excess smoothness in aggregate consumption, notably Flavin (1985), Deaton (1987), and Campbell and Deaton (1989).

A particularly distinctive implication of our model is that responses to labor income shocks differ in direction from responses to shocks in stock returns. Specifically, the model predicts that a negative labor income shock causes households to heavily reduce both savings and risk exposure, whereas a negative stock return shock prompts an increase in both savings and risk exposure. Empirically, one can test for opposite responses by relating observed changes in consumption and portfolio allocations to labor income shocks and stock return shocks.

Our second main implication is that the sensitivity of savings rates and portfolio shares to income shocks depends significantly on the ratio of current consumption to the individual's reference level. Specifically, when consumption is close to the reference level, individuals are predicted to adjust their savings and investment behavior more strongly in response to an income shock. By contrast, when consumption is far from the reference level, these adjustments tend to be more moderate. This prediction can be explored empirically using panel data on savings decisions, portfolio allocations, income, and consumption levels. Since direct data on reference levels are generally unavailable, income can serve as a proxy for the ratio of consumption to the reference level. Indeed, higher income levels are typically associated with a greater gap between consumption and the reference level, while lower incomes suggest that consumption is closer to the reference level. Existing empirical studies support our second main implication; see, e.g., Dynan, Skinner, and Zeldes (2004), Johnson, Parker, and Souleles (2006) and Havránek and Sokolová (2020). Specifically, our results align with the literature documenting that the responsiveness of consumption to income changes depends on the current level of income.

Our third main implication is that an individual with loss-averse reference-dependent preferences who faces labor income risk adopts a more conservative investment strategy at older ages and a less conservative strategy at younger ages. This pattern arises from the interplay between risky labor income, the presence of a reference level, and the individual's flexibility in adjusting that reference level. We can test this hypothesis using data on portfolio holdings and compare the empirical findings with the predictions of standard theory (Bodie et al. (1992)). In addition, a number of studies document limited stock market participation; see, e.g., Mankiw and Zeldes (1991), Vissing-Jørgensen (2002) and Vestman (2019). In our model, the individual's willingness to invest in the stock market declines markedly when labor income uncertainty and loss aversion are high and the reference level is sticky.

5 Conclusion

We have derived and explored the optimal savings and portfolio decisions of an individual with loss-averse reference-dependent preferences who faces risky labor income. We show that both the current optimal savings rate and the current optimal portfolio share are excessively sensitive to labor income shocks under loss-averse reference-dependent preferences, compared to conventional preferences. In the face of severe adverse labor income shocks, a loss-averse individual does not save at all to protect consumption, engages in excessive de-risking, and even withdraws financial wealth prior to retirement.

The response of the savings rate and portfolio share to a permanent labor income shock is strongly influenced by the ratio of consumption to the reference level: if consumption is close to the reference level, the individual drastically reduces her optimal savings rate and risky portfolio share; if consumption is far from the reference level, the response is less pronounced.

Furthermore, the optimal investment strategy is generally conservative at older ages and more aggressive at younger ages compared to conventional preferences. We find that the presence of risky labor income and a reference level typically induces a downward adjustment of the risky asset holding, whereas the flexibility to update the reference level leads to an upward adjustment.

Finally, the loss-averse reference-dependent individual suffers substantial welfare losses when following conventional savings and investment strategies, thus highlighting the importance of suitably designed and sufficiently flexible life-cycle savings and portfolio plans. Indeed, our results have clear practical implications for understanding how investors respond differently to job-related income shocks versus market losses, as well as for the design of age-related investment funds. Institutional constraints can be structured to better align with behavioral responses. In addition, the standard life-cycle recommendation of reducing risk exposure with age is reinforced and, in most cases, should be implemented more aggressively. Importantly, we have evaluated our findings across a range of parameter settings and confirm that the qualitative patterns of behavior and welfare effects remain robust.

A Dual Formulation

In this appendix, we assume a rich specification of the reference level. More specifically, the reference level depends not only on own consumption (as in the main text) but also on individual labor income $Y(t)$ and consumption of the individual's neighbor $\bar{C}(t)$ which is exogenously given. We assume that the reference level can be decomposed as follows:

$$h(t) = h_c(t) + h_{\bar{C}}(t) + h_Y(t). \tag{A1}$$

Here, $h_c(t)$, $h_{\bar{C}}(t)$ and $h_Y(t)$ satisfy the dynamics

$$\mathrm{d}h_c(t) = (\beta_c c(t) - \alpha_c(t)h_c(t))\,\mathrm{d}t, \tag{A2}$$

$$\mathrm{d}h_{\bar{C}}(t) = \big(\beta_{\bar{C}}\bar{C}(t) - \alpha_{\bar{C}}(t)h_{\bar{C}}(t)\big)\,\mathrm{d}t, \tag{A3}$$

$$\mathrm{d}h_Y(t) = (\beta_Y Y(t) - \alpha_Y(t)h_Y(t))\,\mathrm{d}t. \tag{A4}$$

The results in the main text arise as a special case when we assume $h(t) = h_c(t)$, $\beta = \beta_c$ and $\alpha(t) = \alpha_c(t)$.

By Itô's Lemma, we find that the reference level $h(t)$ satisfies

$$\mathrm{d}h(t) = \big(\beta_c c(t) + \beta_{\bar{C}}\bar{C}(t) + \beta_Y Y(t) - \alpha(t)h(t)\big)\,\mathrm{d}t, \tag{A5}$$

where

$$\alpha(t) \equiv \alpha_c(t)\frac{h_c(t)}{h(t)} + \alpha_{\bar{C}}(t)\frac{h_{\bar{C}}(t)}{h(t)} + \alpha_Y(t)\frac{h_Y(t)}{h(t)} \tag{A6}$$

models the rate at which the individual's reference level $h(t)$ depreciates.

In line with the permanent income hypothesis (see Hall (1978)), we assume that changes in consumption of the individual's neighbor are unpredictable. More specifically, we assume that $\bar{C}(t)$ is given by

$$\bar{C}(t) = \bar{C}(0) \exp \left\{ \int_0^t \mu_{\bar{C}} \mathrm{d}s + \sigma_{\bar{C}} \int_0^t \mathrm{d}Z_{\bar{C}}(s) \right\}. \tag{A7}$$

Here, $\mu_{\bar{C}} \in \mathbb{R}$ models the expected growth of $\bar{C}(t)$, $\sigma_{\bar{C}} \geq 0$ represents volatility, and $Z_{\bar{C}}(t)$ is a standard Brownian motion.

We can, by virtue of the martingale approach (Pliska (1986), Karatzas et al. (1987), and Cox and Huang (1989, 1991)), transform the individual's dynamic maximization problem into the following equivalent problem:

$$\begin{aligned}
\max_{c(t)} \quad & \mathbb{E}_0 \left[\int_0^{T_D} e^{-\delta t} u \left(c(t) - h(t) \right) \mathrm{d}t \right] \\
\text{s.t.} \quad & \mathbb{E}_0 \left[\int_0^{T_D} m(t) \left(c(t) - Y(t) \right) \mathrm{d}t \right] \leq F(0), \\
& \mathrm{d}h(t) = \left(\beta_c c(t) + \beta_{\bar{C}} \bar{C}(t) + \beta_Y Y(t) - \alpha(t) h(t) \right) \mathrm{d}t \\
& h(t) \leq c(t) + L(t), \ \text{ for all } t \in [0, T],
\end{aligned} \tag{A8}$$

with $\alpha(t)$, $\bar{C}(t)$, $Y(t)$ and $m(t)$ defined in (A6), (A7), (2.1) and (2.9).

Inspired by Schroder and Skiadas (2002) and Van Bilsen et al. (2020b), we apply the following transformation. Let us denote by $\widehat{c}(t) = c(t) - h(t)$ the individual's dual consumption choice at adult age t. By substituting $c(t) = h(t) + \widehat{c}(t)$ into (A8), we find that (A8) is equivalent to:

$$\begin{aligned}
\max_{\widehat{c}(t)} \quad & \mathbb{E}_0 \left[\int_0^{T_D} e^{-\delta t} u \left(\widehat{c}(t) \right) \mathrm{d}t \right] \\
\text{s.t.} \quad & \mathbb{E}_0 \left[\int_0^{T_D} m(t) \left(h(t) + \widehat{c}(t) - Y(t) \right) \mathrm{d}t \right] \leq F(0), \\
& \mathrm{d}h(t) = \left(\beta_c \widehat{c}(t) + \beta_{\bar{C}} \bar{C}(t) + \beta_Y Y(t) - (\alpha(t) - \beta_c) h(t) \right) \mathrm{d}t \\
& -\widehat{c}(t) \leq L(t), \ \text{ for all } t \in [0, T].
\end{aligned} \tag{A9}$$

By repeated substitution, we are able to derive the analytical form of the reference level. More specifically, we find

$$
\begin{aligned}
h(t) = {} & e^{-\int_0^t(\alpha_c(u)-\beta_c)\mathrm{d}u} h_c(0) + \beta_c \int_0^t e^{-\int_s^t(\alpha_c(u)-\beta_c)\mathrm{d}u} \widehat{c}(s)\mathrm{d}s \\
& + \left[\beta_c \int_0^t e^{-\int_0^s \alpha_{\bar{C}}(u)\mathrm{d}u - \int_s^t(\alpha_c(u)-\beta_c)\mathrm{d}u}\mathrm{d}s + e^{-\int_0^t \alpha_{\bar{C}}(u)\mathrm{d}u}\right] h_{\bar{C}}(0) \\
& + \beta_{\bar{C}} \int_0^t \left[\beta_c \int_s^t e^{-\int_s^v \alpha_{\bar{C}}(u)\mathrm{d}u - \int_v^t(\alpha_c(u)-\beta_c)\mathrm{d}u}\mathrm{d}v + e^{-\int_s^t \alpha_{\bar{C}}(u)\mathrm{d}u}\right] \bar{C}(s)\mathrm{d}s \\
& + \left[\beta_c \int_0^t e^{-\int_0^s \alpha_Y(u)\mathrm{d}u - \int_s^t(\alpha_c(u)-\beta_c)\mathrm{d}u}\mathrm{d}s + e^{-\int_0^t \alpha_Y(u)\mathrm{d}u}\right] h_Y(0) \\
& + \beta_Y \int_0^t \left[\beta_c \int_s^t e^{-\int_s^v \alpha_Y(u)\mathrm{d}u - \int_v^t(\alpha_c(u)-\beta_c)\mathrm{d}u}\mathrm{d}v + e^{-\int_s^t \alpha_Y(u)\mathrm{d}u}\right] Y(s)\mathrm{d}s.
\end{aligned} \tag{A10}
$$

Substitution of (A10) into the static budget constraint in (A9) results in the following new static budget constraint:

$$
\begin{aligned}
F(0) \geq \mathbb{E}_0\Bigg[\int_0^{T_D} & \{f_c(0)h_c(0) + f_{\bar{C}}(0)h_{\bar{C}}(0) + f_Y(0)h_Y(0) + m(t)\left(1+\beta_c f_c(t)\right)\widehat{c}(t) \\
& + \beta_{\bar{C}} f_{\bar{C}}(t)\bar{C}(t)m(t) - Y(t)m(t)\left(1-\beta_Y f_Y(t)\right)\}\,\mathrm{d}t\Bigg].
\end{aligned} \tag{A11}
$$

Here,

$$
f_c(t) \equiv \mathbb{E}_t\left[\int_t^{T_D} \frac{m(s)}{m(t)} e^{-\int_t^s(\alpha_c(u)-\beta_c)\mathrm{d}u}\mathrm{d}s\right], \tag{A12}
$$

$$
\begin{aligned}
f_{\bar{C}}(t) \equiv \mathbb{E}_t\Bigg[\int_t^{T_D} \frac{m(s)}{m(t)} \Bigg\{\beta_c \int_t^s & e^{-\int_t^v \alpha_{\bar{C}}(u)\mathrm{d}u - \int_v^s(\alpha_c(u)-\beta_c)\mathrm{d}u}\mathrm{d}v \\
& + e^{-\int_t^s \alpha_{\bar{C}}(u)\mathrm{d}u}\Bigg\}\mathrm{d}s\Bigg],
\end{aligned} \tag{A13}
$$

$$
\begin{aligned}
f_Y(t) \equiv \mathbb{E}_t\Bigg[\int_t^{T_D} \frac{m(s)}{m(t)} \Bigg\{\beta_c \int_t^s & e^{-\int_t^v \alpha_Y(u)\mathrm{d}u - \int_v^s(\alpha_c(u)-\beta_c)\mathrm{d}u}\mathrm{d}v \\
& + e^{-\int_t^s \alpha_Y(u)\mathrm{d}u}\Bigg\}\mathrm{d}s\Bigg].
\end{aligned} \tag{A14}
$$

We denote by $\widehat{m}(t)$, $\widehat{Y}(t)$ and $\widehat{F}(t)$ the dual counterparts of the pricing kernel, individual labor income and financial wealth, respectively. These variables are defined as follows:

$$\widehat{m}(t) \equiv m(t)\left(1+\beta_c f_c(t)\right), \tag{A15}$$

$$\widehat{Y}(t) \equiv \frac{Y(t)\left(1-\beta_Y f_Y(t)\right)-\beta_{\bar{C}} f_{\bar{C}}(t)\bar{C}(t)}{1+\beta_c f_c(t)}, \tag{A16}$$

$$\widehat{F}(t) \equiv \frac{F(t)-f_c(t)h_c(t)-f_{\bar{C}}(t)h_{\bar{C}}(t)-f_Y(t)h_Y(t)}{1+\beta_c f_c(t)}. \tag{A17}$$

We can now write the new static budget constraint (A11) in familiar form as:

$$\mathbb{E}_0\left[\int_0^{T_D} \frac{\widehat{m}(t)}{\widehat{m}(0)}\left(\widehat{c}(t)-\widehat{Y}(t)\right)\mathrm{d}t\right] \leq \widehat{F}(0). \tag{A18}$$

Hence, the individual maximizes the following static dual maximization problem:

$$\begin{aligned}
\max_{\widehat{c}(t)} \quad & \mathbb{E}_0\left[\int_0^{T_D} e^{-\delta t} u\left(\widehat{c}(t)\right)\mathrm{d}t\right] \\
\text{s.t.} \quad & \mathbb{E}_0\left[\int_0^{T_D} \frac{\widehat{m}(t)}{\widehat{m}(0)}\left(\widehat{c}(t)-\widehat{Y}(t)\right)\mathrm{d}t\right] \leq \widehat{F}(0) \\
& -\widehat{c}(t) \leq L(t), \text{ for all } t \in [0,T].
\end{aligned} \tag{A19}$$

Problem (A19) is typically easier to solve than the original individual's dynamic maximization problem (2.8). We note that the dual pricing kernel $\widehat{m}(t) \equiv m(t)\left(1+\beta_c f_c(t)\right)$ satisfies the following dynamics:

$$\frac{\mathrm{d}\widehat{m}(t)}{\widehat{m}(t)} = -\widehat{r}(t)\mathrm{d}t + \widehat{\phi}(t)^\top \mathrm{d}Z(t), \tag{A20}$$

with

$$\widehat{\phi}(t) = \phi(t), \tag{A21}$$

$$\widehat{r}(t) \equiv \beta_c + \frac{r-\alpha_c(t)\beta_c f_c(t)}{1+\beta_c f_c(t)}. \tag{A22}$$

B Derivation of Optimal Policies

B.1 Derivation of Optimal Consumption Choice

Let us first assume that the financial market is complete (i.e., aggregate consumption risk and individual labor income risk are tradable), so that $\phi(t)$ is uniquely determined. We now determine the individual's optimal consumption choice given this assumption. The Lagrangian $\mathcal{L}$ is given by

$$\mathcal{L} = \mathbb{E}_0\left[\int_0^{T_D} e^{-\delta t} u\left(\widehat{c}(t)\right) \mathrm{d}t\right] + y\left(\widehat{F}(0) - \mathbb{E}_0\left[\int_0^{T_D} \frac{\widehat{m}(t)}{\widehat{m}(0)}\left(\widehat{c}(t) - \widehat{Y}(t)\right) \mathrm{d}t\right]\right). \qquad \text{(A1)}$$

Here, $y \geq 0$ denotes the Lagrange multiplier associated with the static budget constraint. For every t, the individual aims to maximize $e^{-\delta t}u\left(\widehat{c}(t)\right) - y\widehat{m}(t)\widehat{c}(t)/\widehat{m}(0)$ subject to $-\widehat{c}(t) \leq L(t)$. Denote by u_G the part of the utility function above the reference level, and by u_L the part of the utility function below the reference level. Let $c_G^+(t)$ and $c_L^+(t)$ be the optimal solutions of the functions u_G and u_L, respectively.

We first consider the case where the individual is risk averse in the loss domain. Due to the concavity of u_G and u_L, the optimal dual consumption choices $c_G^+(t)$ and $c_L^+(t)$ satisfy the following optimality conditions (for $j = G, L$):[27]

$$e^{-\delta t} u_j'\left(c_j^+(t)\right) = y\widehat{m}(t)/\widehat{m}(0) - x_j(t), \qquad \text{(A2)}$$

$$-c_j^+(t) \leq L(t), \qquad \text{(A3)}$$

$$x_j(t)\left(c_j^+(t) + L(t)\right) = 0, \qquad \text{(A4)}$$

$$x_j(t) \geq 0. \qquad \text{(A5)}$$

Here, $x_j(t)$ $(j = G, L)$ denotes the Lagrange multiplier associated with the constraint on dual consumption. After solving the optimality conditions, we obtain the following two

[27]We denote the derivative of a function f at a point a by $f'(a)$.

local maxima:

$$c_G^+(t) = \left(\frac{ye^{\delta t}\widehat{m}(t)}{\gamma_G \widehat{m}(0)}\right)^{\frac{1}{\gamma_G - 1}}, \tag{A6}$$

$$c_L^+(t) = -\min\left\{\left(\frac{ye^{\delta t}\widehat{m}(t)}{\eta\gamma_L \widehat{m}(0)}\right)^{\frac{1}{\gamma_L - 1}}, L(t)\right\}. \tag{A7}$$

To determine the global maximum $c^+(t)$, we introduce the following function:

$$\begin{aligned} f\left(\frac{\widehat{m}(t)}{\widehat{m}(0)}\right) &= e^{-\delta t} u\left(c_G^+(t)\right) - y\frac{\widehat{m}(t)}{\widehat{m}(0)} c_G^+(t) - \left(e^{-\delta t} u\left(c_L^+(t)\right) - y\frac{\widehat{m}(t)}{\widehat{m}(0)} c_L^+(t)\right) \\ &= e^{-\delta t}\left(1-\gamma_G\right)\left(\frac{ye^{\delta t}\widehat{m}(t)}{\gamma_G \widehat{m}(0)}\right)^{\frac{\gamma_G}{\gamma_G - 1}} \\ &\quad + e^{-\delta t}\eta \min\left\{\left(\frac{ye^{\delta t}\widehat{m}(t)}{\eta\gamma_L \widehat{m}(0)}\right)^{\frac{1}{\gamma_L - 1}}, L(t)\right\}^{\gamma_L} \\ &\quad - y\frac{\widehat{m}(t)}{\widehat{m}(0)} \max\left\{\left(\frac{ye^{\delta t}\widehat{m}(t)}{\eta\gamma_L \widehat{m}(0)}\right)^{\frac{1}{\gamma_L - 1}}, L(t)\right\}. \end{aligned} \tag{A8}$$

The global maximum $c^+(t)$ is equal to $c_G^+(t)$ if $f\left(\frac{\widehat{m}(t)}{\widehat{m}(0)}\right) \geq 0$; and equals $c_L^+(t)$ otherwise. It follows that $\lim_{\frac{\widehat{m}(t)}{\widehat{m}(0)}\to\infty} f\left(\frac{\widehat{m}(t)}{\widehat{m}(0)}\right) = -\infty$, $\lim_{\frac{\widehat{m}(t)}{\widehat{m}(0)}\to 0} f\left(\frac{\widehat{m}(t)}{\widehat{m}(0)}\right) = \infty$ and $f'\left(\frac{\widehat{m}(t)}{\widehat{m}(0)}\right) < 0$ for all $\frac{\widehat{m}(t)}{\widehat{m}(0)}$. Hence, $f\left(\frac{\widehat{m}(t)}{\widehat{m}(0)}\right)$ is strictly decreasing. As a result, $f\left(\frac{\widehat{m}(t)}{\widehat{m}(0)}\right)$ has one zero in the interval $(0,\infty)$. We define $\xi(t)$ such that $f\left(\xi(t)\right) = 0$. The global maximum $c^+(t)$ is equal to $c_G^+(t)$ if $\frac{\widehat{m}(t)}{\widehat{m}(0)} \leq \xi(t)$; and equals $c_L^+(t)$ otherwise.

We now consider the case where the individual is risk loving in the loss domain. Due to the concavity of u_G, the optimal dual consumption choice $c_G^+(t)$ satisfies the following optimality conditions:

$$e^{-\delta t} v'_G \left(c_G^+(t)\right) = y\widehat{m}(t)/\widehat{m}(0) - x_G(t), \tag{A9}$$

$$-c_G^+(t) \leq L(t), \tag{A10}$$

$$x_G(t) \left(c_G^+(t) + L(t)\right) = 0, \tag{A11}$$

$$x_G(t) \geq 0. \tag{A12}$$

After solving the optimality conditions, we obtain the following local maximum:

$$c_G^+(t) = \left(\frac{ye^{\delta t}\widehat{m}(t)}{\gamma_G \widehat{m}(0)}\right)^{\frac{1}{\gamma_G - 1}}.$$

Due to the convexity of u_L, the optimal dual consumption choice $c_L^+(t)$ lies at a corner point of the feasible region. Hence, the only two possible candidates for $c_L^+(t)$ are $-L(t)$ and 0.

To determine the global maximum $c^+(t)$, we introduce the following function:

$$g\left(\frac{\widehat{m}(t)}{\widehat{m}(0)}\right) = e^{-\delta t} u\left(c_G^+(t)\right) - y\frac{\widehat{m}(t)}{\widehat{m}(0)} c_G^+(t) - \left(e^{-\delta t} u\left(c_L^+(t)\right) - y\frac{\widehat{m}(t)}{\widehat{m}(0)} c_L^+(t)\right). \tag{A13}$$

The global maximum $c^+(t)$ is equal to $c_G^+(t)$ if $g\left(\frac{\widehat{m}(t)}{\widehat{m}(0)}\right) \geq 0$; and equals $c_L^+(t)$ otherwise. We distinguish between the following two cases:

- $c_L^+(t) = 0$. Straightforward computations show that $g\left(\frac{\widehat{m}(t)}{\widehat{m}(0)}\right)$ is given by

$$g\left(\frac{\widehat{m}(t)}{\widehat{m}(0)}\right) = e^{-\delta t}\left(1 - \gamma_G\right)\left(\frac{ye^{\delta t}\widehat{m}(t)}{\gamma_G \widehat{m}(0)}\right)^{\frac{\gamma_G}{\gamma_G - 1}}. \tag{A14}$$

  Since $0 < \gamma_G < 1$ and $y > 0$, it follows that $g\left(\frac{\widehat{m}(t)}{\widehat{m}(0)}\right) > 0$ for all $\frac{\widehat{m}(t)}{\widehat{m}(0)}$. We conclude that $c_L^+(t) = 0$ is never optimal.

- $c_L^+(t) = -L(t)$. Straightforward computations show that $g\left(\frac{\widehat{m}(t)}{\widehat{m}(0)}\right)$ is given by

$$g\left(\frac{\widehat{m}(t)}{\widehat{m}(0)}\right) = e^{-\delta t}\left(1-\gamma_G\right)\left(\frac{ye^{\delta t}\widehat{m}(t)}{\gamma_G\widehat{m}(0)}\right)^{\frac{\gamma_G}{\gamma_G-1}} + e^{-\delta t}\eta L(t)^{\gamma_L} - y\frac{\widehat{m}(t)}{\widehat{m}(0)}L(t). \quad \text{(A15)}$$

  It follows that $g\left(\frac{\widehat{m}(t)}{\widehat{m}(0)}\right) > 0$ for all $\frac{\widehat{m}(0)}{\widehat{m}(0)} \leq \frac{\eta}{y}e^{-\delta t}L(t)^{\gamma_L-1}$. Also, $\lim_{\frac{\widehat{m}(t)}{\widehat{m}(0)}\to\infty} g\left(\frac{\widehat{m}(t)}{\widehat{m}(0)}\right) = -\infty$ and $g'\left(\frac{\widehat{m}(t)}{\widehat{m}(0)}\right) < 0$ for all $\frac{\widehat{m}(t)}{\widehat{m}(0)}$. Hence, $g\left(\frac{\widehat{m}(t)}{\widehat{m}(0)}\right)$ is strictly decreasing. As a result, $g\left(\frac{\widehat{m}(t)}{\widehat{m}(0)}\right)$ has one zero in the interval $\left(\frac{\eta}{y}e^{-\delta t}L(t)^{\gamma_L-1}, \infty\right)$. We define $\zeta(t)$ such that $g\left(\zeta(t)\right) = 0$. It follows that the global maximum $c^+(t)$ is equal to $c_G^+(t)$ if $\frac{\widehat{m}(t)}{\widehat{m}(0)} \leq \zeta(t)$; and equals $c_L^+(t)$ otherwise.

We observe from the analysis above that, in both cases, the global maximum $c^+(t)$ is a function of $\widehat{m}(t)/\widehat{m}(0)$, i.e., $c^+(t) = h\left(\frac{\widehat{m}(t)}{\widehat{m}(0)}\right)$ for some function h.

Let us now assume that the market is incomplete (i.e., aggregate consumption risk and individual labor income risk are *not* tradable), so that $\phi(t)$ is not uniquely determined. In line with, e.g., He and Pearson (1991), we determine the vector of factor loadings such that changes in dual total wealth match changes in the value of future dual consumption; see Appendix B.2 and the online appendix for more details. We denote the vector of factor loadings that satisfies this condition by $\phi^*(t) = \widehat{\phi}^*(t)$. The pricing kernel implied by $\phi^*(t)$ is denoted by $m^*(t)$. It now follows that

$$\widehat{c}^*(t) = h\left(\frac{\widehat{m}^*(t)}{\widehat{m}^*(0)}\right) \quad \text{(A16)}$$

not only maximizes the individual's static maximization problem (A19) but also can be replicated in our financial market as defined in Section 2.1.

The optimal (primal) consumption choice is given by

$$c^*(t) = h^*(t) + \widehat{c}^*(t), \quad \text{(A17)}$$

with the optimal reference level $h^*(t)$ defined as follows:

$$
\begin{aligned}
h^*(t) = {} & e^{-\int_0^t(\alpha_c(u)-\beta_c)\mathrm{d}u} h_c(0) + \beta_c \int_0^t e^{-\int_s^t(\alpha_c(u)-\beta_c)\mathrm{d}u} \widehat{c}^*(s)\mathrm{d}s \\
& + \left[\beta_c \int_0^t e^{-\int_0^s \alpha_{\bar{C}}(u)\mathrm{d}u - \int_s^t(\alpha_c(u)-\beta_c)\mathrm{d}u}\mathrm{d}s + e^{-\int_0^t \alpha_{\bar{C}}(u)\mathrm{d}u}\right] h_{\bar{C}}(0) \\
& + \beta_{\bar{C}} \int_0^t \left[\beta_c \int_s^t e^{-\int_s^v \alpha_{\bar{C}}(u)\mathrm{d}u - \int_v^t(\alpha_c(u)-\beta_c)\mathrm{d}u}\mathrm{d}v + e^{-\int_s^t \alpha_{\bar{C}}(u)\mathrm{d}u}\right] \bar{C}(s)\mathrm{d}s \\
& + \left[\beta_c \int_0^t e^{-\int_0^s \alpha_Y(u)\mathrm{d}u - \int_s^t(\alpha_c(u)-\beta_c)\mathrm{d}u}\mathrm{d}s + e^{-\int_0^t \alpha_Y(u)\mathrm{d}u}\right] h_Y(0) \\
& + \beta_Y \int_0^t \left[\beta_c \int_s^t e^{-\int_s^v \alpha_Y(u)\mathrm{d}u - \int_v^t(\alpha_c(u)-\beta_c)\mathrm{d}u}\mathrm{d}v + e^{-\int_s^t \alpha_Y(u)\mathrm{d}u}\right] Y(s)\mathrm{d}s.
\end{aligned} \tag{A18}
$$

B.2 Derivation of Optimal Portfolio Choice and Optimal Vector of Factor Loadings

We determine the dual portfolio strategy $\widehat{\omega}(t)$ and the vector of dual factor loadings $\widehat{\phi}(t) = -\rho^{-1}\widehat{\lambda}(t)$ such that changes in dual total wealth $\mathrm{d}\widehat{W}(t)$ match changes in the value of future optimal dual consumption $\mathrm{d}\widehat{V}(t)$ where the value of future optimal dual consumption is defined as follows:

$$
\widehat{V}(t) \equiv \mathbb{E}_t \left[\int_t^{T_D} \frac{\widehat{m}(s)}{\widehat{m}(t)} h\left(\frac{\widehat{m}(s)}{\widehat{m}(0)}\right)\mathrm{d}s\right] = f_{\widehat{V}}(t, \widehat{Y}(t), \widehat{m}(t)). \tag{A19}
$$

We note that the individual's dual total wealth $\widehat{W}(t)$ is the sum of the individual's dual financial wealth $\widehat{F}(t)$ and the individual's dual human wealth $\widehat{H}(t)$, where $\widehat{H}(t)$ is defined as follows:

$$
\widehat{H}(t) \equiv \mathbb{E}_t \left[\int_t^{T_D} \frac{\widehat{m}(s)}{\widehat{m}(t)} \widehat{Y}(s)\mathrm{d}s\right] = f_{\widehat{H}}\left(t, \widehat{Y}(t), \widehat{m}(t)\right). \tag{A20}
$$

Hence, to derive the dynamics of $\widehat{W}(t)$, we first need to derive the dynamics of $\widehat{F}(t)$ and $\widehat{H}(t)$. By Itô's Lemma, we find

$$\begin{aligned}
\mathrm{d}\widehat{H}(t) = (\ldots)\,\mathrm{d}t &+ \left(\frac{\partial \widehat{H}(t)}{\partial \widehat{Y}(t)}\frac{\partial \widehat{Y}(t)}{\partial \bar{C}(t)}\sigma_{\bar{C}}\bar{C}(t) + \frac{\partial \widehat{H}(t)}{\partial \widehat{m}(t)}\widehat{\phi}_{\bar{C}}(t)\widehat{m}(t)\right)\mathrm{d}Z_{\bar{C}}(t) \\
&+ \left(\frac{\partial \widehat{H}(t)}{\partial \widehat{Y}(t)}\frac{\partial \widehat{Y}(t)}{\partial Y(t)}\sigma_{Y}Y(t) + \frac{\partial \widehat{H}(t)}{\partial \widehat{m}(t)}\widehat{\phi}_{Y}(t)\widehat{m}(t)\right)\mathrm{d}Z_{Y}(t) \\
&+ \frac{\partial \widehat{H}(t)}{\partial \widehat{m}(t)}\widehat{\phi}_{S}(t)\mathrm{d}Z_{S}(t).
\end{aligned} \tag{A21}$$

Here,

$$\widehat{\phi}_S(t) = -\widehat{\lambda}_S - \rho^{-1}_{2,1}\widehat{\lambda}_{\bar{C}}(t) - \rho^{-1}_{3,1}\widehat{\lambda}_Y(t), \tag{A22}$$

$$\widehat{\phi}_{\bar{C}}(t) = -\rho^{-1}_{1,2}\widehat{\lambda}_S - \widehat{\lambda}_{\bar{C}}(t) - \rho^{-1}_{3,2}\widehat{\lambda}_Y(t), \tag{A23}$$

$$\widehat{\phi}_Y(t) = -\rho^{-1}_{1,3}\widehat{\lambda}_S - \rho^{-1}_{2,3}\widehat{\lambda}_{\bar{C}}(t) - \widehat{\lambda}_Y(t), \tag{A24}$$

with $\rho^{-1}_{i,j}$ the (i,j)th element of ρ^{-1}. Note that $\widehat{\lambda}_S = \lambda_S = (\mu_S - r)/\sigma_S$.

The individual's dual financial wealth $\widehat{F}(t)$ evolves as follows:

$$\mathrm{d}\widehat{F}(t) = (\ldots)\,\mathrm{d}t + \widehat{\omega}(t)\sigma_S\widehat{F}(t)\mathrm{d}Z_S(t), \tag{A25}$$

where $\widehat{\omega}(t)$ represents the share of dual financial wealth invested in the risky stock.

Thus,

$$\begin{aligned} \mathrm{d}\widehat{W}(t) &= \mathrm{d}\widehat{F}(t) + \mathrm{d}\widehat{H}(t) \\ &= (\ldots)\,\mathrm{d}t + \left(\frac{\partial \widehat{H}(t)}{\partial \widehat{Y}(t)} \frac{\partial \widehat{Y}(t)}{\partial \bar{C}(t)} \sigma_{\bar{C}} \bar{C}(t) + \frac{\partial \widehat{H}(t)}{\partial \widehat{m}(t)} \widehat{\phi}_{\bar{C}}(t) \widehat{m}(t) \right) \mathrm{d}Z_{\bar{C}}(t) \\ &\quad + \left(\frac{\partial \widehat{H}(t)}{\partial \widehat{Y}(t)} \frac{\partial \widehat{Y}(t)}{\partial Y(t)} \sigma_Y Y(t) + \frac{\partial \widehat{H}(t)}{\partial \widehat{m}(t)} \widehat{\phi}_Y(t) \widehat{m}(t) \right) \mathrm{d}Z_Y(t) \\ &\quad + \left(\widehat{\omega}(t) \sigma_S \widehat{F}(t) + \frac{\partial \widehat{H}(t)}{\partial \widehat{m}(t)} \widehat{\phi}_S(t) \right) \mathrm{d}Z_S(t). \end{aligned} \tag{A26}$$

By Itô's Lemma, we also find

$$\begin{aligned} \mathrm{d}\widehat{V}(t) &= (\ldots)\,\mathrm{d}t + \left(\frac{\partial \widehat{V}(t)}{\partial \widehat{Y}(t)} \frac{\partial \widehat{Y}(t)}{\partial \bar{C}(t)} \sigma_{\bar{C}} \bar{C}(t) + \frac{\partial \widehat{V}(t)}{\partial \widehat{m}(t)} \widehat{\phi}_{\bar{C}}(t) \widehat{m}(t) \right) \mathrm{d}Z_{\bar{C}}(t) \\ &\quad + \left(\frac{\partial \widehat{V}(t)}{\partial \widehat{Y}(t)} \frac{\partial \widehat{Y}(t)}{\partial Y(t)} \sigma_Y Y(t) + \frac{\partial \widehat{V}(t)}{\partial \widehat{m}(t)} \widehat{\phi}_Y(t) \widehat{m}(t) \right) \mathrm{d}Z_Y(t) \\ &\quad + \frac{\partial \widehat{V}(t)}{\partial \widehat{m}(t)} \widehat{\phi}_S(t) \widehat{m}(t) \mathrm{d}Z_S(t). \end{aligned} \tag{A27}$$

Solving $\mathrm{d}\widehat{V}(t) = \mathrm{d}\widehat{W}(t)$, we find that the optimal dual portfolio strategy and the optimal dual factor loadings should satisfy

$$\widehat{\omega}^*(t)\sigma_S \widehat{F}(t) = \frac{\partial \widehat{F}(t)}{\partial \widehat{m}(t)} \widehat{\phi}^*_S(t) \widehat{m}(t), \tag{A28}$$

$$\widehat{\phi}^*_{\bar{C}}(t) = \left(\frac{\partial \widehat{H}(t)}{\partial \bar{C}(t)} - \frac{\partial \widehat{V}(t)}{\partial \bar{C}(t)} \right) \left(\frac{\partial \widehat{V}(t)}{\partial \widehat{m}(t)} - \frac{\partial \widehat{H}(t)}{\partial \widehat{m}(t)} \right)^{-1} \frac{\sigma_{\bar{C}} \bar{C}(t)}{\widehat{m}(t)}, \tag{A29}$$

$$\widehat{\phi}^*_Y(t) = \left(\frac{\partial \widehat{H}(t)}{\partial Y(t)} - \frac{\partial \widehat{V}(t)}{\partial Y(t)} \right) \left(\frac{\partial \widehat{V}(t)}{\partial \widehat{m}(t)} - \frac{\partial \widehat{H}(t)}{\partial \widehat{m}(t)} \right)^{-1} \frac{\sigma_Y Y(t)}{\widehat{m}(t)}. \tag{A30}$$

The online appendix describes how to numerically determine $\widehat{\omega}^*(t)$, $\widehat{\lambda}^*_{\bar{C}}(t)$ and $\widehat{\lambda}^*_Y(t)$.

Note that $F(t) = f_c(t)h_c(t) + f_{\bar{C}}(t)h_{\bar{C}}(t) + f_Y(t)h_Y(t) + \widehat{F}(t)\left(1 + \beta_c f_c(t)\right)$. Hence, by

Itô's Lemma,

$$\mathrm{d}F(t) = (\ldots)\,\mathrm{d}t + (1 + \beta_c f_c(t))\,\widehat{\omega}(t)\sigma_S \widehat{F}(t)\mathrm{d}Z_S(t). \tag{A31}$$

Comparing (A31) with the dynamic budget constraint

$$\mathrm{d}F(t) = (\ldots)\,\mathrm{d}t + \omega(t)\sigma_S F(t)\mathrm{d}Z_S(t), \tag{A32}$$

we find that the optimal portfolio strategy $\omega^*(t)$ is given by

$$\omega^*(t) = \widehat{\omega}^*(t)\,(1 + \beta_c f_c(t))\,\frac{\widehat{F}(t)}{F(t)}. \tag{A33}$$

References

Abel, A. B. 1990. Asset Prices under Habit Formation and Catching up with the Joneses. *The American Economic Review* 80:38–42.

Ameriks, J., and S. Zeldes. 2004. How Do Household Portfolio Shares Vary with Age? Working Paper.

Andries, M. 2012. Consumption-Based Asset Pricing with Loss Aversion. Working Paper.

Barberis, N., M. Huang, and T. Santos. 2001. Prospect Theory and Asset Prices. *The Quarterly Journal of Economics* 116:1–53.

Barberis, N., M. Huang, and R. H. Thaler. 2006. Individual Preferences, Monetary Gambles, and Stock Market Participation: A Case for Narrow Framing. *American Economic Review* 96:1069–1090.

Benzoni, L., P. Collin-Dufresne, and R. S. Goldstein. 2007. Portfolio Choice over the Life-Cycle when the Stock and Labor Markets Are Cointegrated. *Journal of Finance* 62:2123–2167.

Berkelaar, A. B., R. Kouwenberg, and T. Post. 2004. Optimal Portfolio Choice under Loss Aversion. *The Review of Economics and Statistics* 86:973–987.

Bodie, Z., R. C. Merton, and W. F. Samuelson. 1992. Labor Supply Flexibility and Portfolio Choice in a Life Cycle Model. *Journal of Economic Dynamics and Control* 16:427–449.

Campbell, J. Y., J. Cocco, F. Gomes, P. J. Maenhout, and L. M. Viceira. 2001. Stock Market Mean Reversion and the Optimal Equity Allocation of a Long-Lived Investor. *European Finance Review* 5:269–292.

Campbell, J. Y., and J. Cochrane. 1999. By Force of Habit: A Consumption-Based Explanation of Aggregate Stock Market Behavior. *Journal of Political Economy* 107:205–251.

Campbell, J. Y., and A. Deaton. 1989. Why is Consumption so Smooth? *Review of Economic Studies* 56:357–373.

Caroll, C. D. 1997. Buffer-Stock Saving and the Life Cycle/Permanent Income Hypothesis. *Quarterly Journal of Economics* 112:1–55.

Caroll, C. D., and A. A. Samwick. 1997. The Nature of Precautionary Wealth. *Journal of Monetary Economics* 40:41–71.

Chacko, G., and L. M. Viceira. 2005. Dynamic Consumption and Portfolio Choice with Stochastic Volatility in Incomplete Markets. *The Review of Financial Studies* 18:1369–1402.

Chai, J., W. J. Horneff, R. H. Maurer, and O. S. Mitchell. 2011. Optimal Portfolio Choice over the Life Cycle with Flexible Work, Endogenous Retirement, and Lifetime Payouts. *Review of Finance* 15:875–907.

Chamberlain, G., and K. Hirano. 1999. Predictive Distributions Based on Longitudinal Earnings Data. *Annales d'Economie et de Statistique* 55-56:211–242.

Chen, A. Y. 2017. External Habit in a Production Economy: A Model of Asset Prices and Consumption Volatility Risk. *The Review of Financial Studies* 30:2890–2932.

Cocco, J. F., F. J. Gomes, and P. J. Maenhout. 2005. Consumption and Portfolio Choice over the Life Cycle. *The Review of Financial Studies* 18:491–533.

Constantinides, G. M. 1990. Habit Formation: A Resolution of the Equity Premium Puzzle. *Journal of Political Economy* 98:519–543.

Cox, J. C., and C. Huang. 1989. Optimal Consumption and Portfolio Policies when Asset Prices Follow a Diffusion Process. *Journal of Economic Theory* 49:33–83.

Cox, J. C., and C. Huang. 1991. A Variational Problem Arising in Financial Economics. *Journal of Mathematical Economics* 20:465–487.

Curatola, G. 2017. Optimal Portfolio Choice with Loss Aversion over Consumption. *Quarterly Review of Economics and Finance* 66:345–358.

de Jong, F. 2008. Valuation of Pension Liabilities in Incomplete Markets. *Journal of Pension Economics and Finance* 7:277–294.

Deaton, A. 1987. Life-Cycle Models of Consumption: Is the Evidence Consistent with the Theory? In T. F. Bewley (ed.), *Advances in econometrics: fifth world congress*, vol. 2, pp. 121–148.

Deng, S., X. Li, H. Pham, and X. Yu. 2022. Optimal Consumption with Reference to Past Spending Maximum. *Finance and Stochastics* 26:217–266.

Detemple, J. B., and F. Zapatero. 1991. Asset Prices in an Exchange Economy with Habit Formation. *Econometrica* 59:1633–1657.

Dynan, K. E., J. Skinner, and S. P. Zeldes. 2004. Do the Rich Save More? *Journal of Political Economy* 112:397–444.

Flavin, M. 1985. Excess Sensitivity of Consumption to Current Income: Liquidity Constraints or Myopia? *Canadian Journal of Economics* 18:117–136.

Gomes, F., and A. Michaelides. 2003. Portfolio Choice with Internal Habit Formation: a Life-Cycle Model with Uninsurable Labor Income Risk. *Review of Economic Dynamics* 6:729–766.

Gomes, F. J. 2005. Portfolio Choice and Trading Volume with Loss Averse Investors. *The Journal of Business* 78:675–706.

Gomes, F. J., L. J. Kotlikoff, and L. M. Viceira. 2008. Optimal Life-Cycle Investing with Flexible Labor Supply: A Welfare Analysis of Life-Cycle Funds. *American Economic Review* 98:297–303.

Guasoni, P., G. Huberman, and D. Ren. 2020. Shortfall Aversion. *Mathematical Finance* 30:869–920.

Hall, R. E. 1978. Stochastic Implications of the Life Cycle-Permanent Income Hypothesis: Theory and Evidence. *Journal of Political Economy* 86:971–987.

Havránek, T., and M. Sokolová. 2020. Do Consumers Really Follow a Rule of Thumb? Three Thousand Estimates from 144 Studies Say "Probably Not". *Review of Economic Dynamics* 35:97–122.

He, H., and N. Pearson. 1991. Consumption and Portfolio Policies With Incomplete Markets and Short-Sale Constraints: The Infinite Dimensional Case. *Journal of Economic Theory* 54:259–304.

He, X. D., and M. S. Strub. 2022. How Endogenization of the Reference Point Affects Loss Aversion: A Study of Portfolio Selection. *Operations Research* 70:3035–3053.

He, X. D., and X. Y. Zhou. 2011. Portfolio Choice via Quantiles. *Mathematical Finance* 21:203–231.

He, X. D., and X. Y. Zhou. 2016. Hope, Fear and Aspirations. *Mathematical Finance* 26:3–50.

Hubener, A., R. Maurer, and O. S. Mitchell. 2016. How Family Status and Social Security Claiming Options Shape Optimal Life Cycle Portfolios. *Review of Finance* 29:937–978.

Jin, H., and X. Y. Zhou. 2008. Behavioral Portfolio Selection in Continuous Time. *Mathematical Finance* 18:385–426.

Johnson, D. S., J. A. Parker, and N. S. Souleles. 2006. Household Expenditure and the Income Tax Rebates of 2001. *American Economic Review* 5:1589–1610.

Jorgenson, D. W., and B. M. Fraumeni. 1989. The Accumulation of Human and Non-Human Capital. In R. Lipsey and H. S. Tice (eds.), *The Measurement of Saving, Investment, and Wealth*, pp. 227–286. Chicago: University of Chicago Press.

Kahneman, D., and A. Tversky. 1979. Prospect Theory: An Analysis of Decision under Risk. *Econometrica* 47:263–292.

Karatzas, I., J. P. Lehoczky, and S. E. Shreve. 1987. Optimal Consumption and Portfolio Decisions for a "Small Investor" on a Finite Horizon. *SIAM Journal of Control and Optimization* 25:1557–1586.

Karatzas, I., and S. E. Shreve. 1998. *Methods of Mathematical Finance (Stochastic Modelling and Applied Probability)*, vol. 39. New York: Springer.

Kőszegi, B., and M. Rabin. 2006. A Model of Reference-Dependent Preferences. *The Quarterly Journal of Economics* 121:1133–1165.

Laeven, R. J. A., and M. A. Stadje. 2014. Robust Portfolio Choice and Indifference Valuation. *Mathematics of Operations Research* 39:1109–1141.

Langtry, A. 2023. Keeping Up with "The Joneses": Reference-Dependent Choice with Social Comparisons. *American Economic Journal: Microeconomics* 15:474–500.

Liang, Z., X. Luo, and F. Yuan. 2023. Consumption-Investment Decisions with Endogenous Reference Point and Drawdown Constraint. *Mathematics and Financial Economics* 17:285–334.

Liu, J. 2007. Portfolio Selection in Stochastic Environments. *The Review of Financial Studies* 20:1–39.

Liu, X., X. Yu, and Q. Zhang. 2024. Optimal Consumption with Loss Aversion and Reference to Past Spending Maximum. *SIAM Journal on Financial Mathematics* 15:121–160.

Lustig, H., S. van Nieuwerburgh, and A. Verdelhan. 2013. The Wealth-Consumption Ratio. *Review of Asset Pricing Studies* 3:38–94.

Mankiw, N. G., and S. P. Zeldes. 1991. The Consumption of Stockholders and Nonstockholders. *Journal of Financial Economics* 29:97–112.

Maurer, R. H., O. S. Mitchell, R. Rogalla, and T. Schimetschek. 2021. Optimal Social Security Claiming Behavior under Lump Sum Incentives: Theory and Evidence. *Journal of Risk and Insurance* 88:5–27.

Mayers, D. 1972. Nonmarketable Assets and Capital Market Equilibrium under Uncertainty. In M. C. Jensen (ed.), *Studies in the Theory of Capital Markets*, pp. 223–248. New York: Praeger Publishers Inc.

Merton, R. C. 1969. Lifetime Portfolio Selection under Uncertainty: The Continuous-Time Case. *The Review of Economics and Statistics* 51:247–257.

Mossin, J. 1968. Optimal Multiperiod Portfolio Policies. *Journal of Business* 41:215–229.

Munk, C. 2008. Portfolio and Consumption Choice with Stochastic Investment Opportunities and Habit formation in Preferences. *Journal of Economic Dynamics and Control* 32:3560–3589.

Pagel, M. 2016. Expectations-Based Reference-Dependent Preferences and Asset Pricing. *Journal of the European Economic Association* 14:468–514.

Pagel, M. 2017. Expectations-Based Reference-Dependent Life-Cycle Consumption. *Review of Economic Studies* 84:885–934.

Pagel, M. 2018. A News-Utility Theory for Inattention and Delegation in Portfolio Choice. *Econometrica* 86:491–522.

Pliska, S. R. 1986. A Stochastic Calculus Model of Continuous Trading: Optimal Portfolios. *Mathematics of Operations Research* 11:371–382.

Samuelson, P. A. 1969. Lifetime Portfolio Selection by Dynamic Stochastic Programming. *The Review of Economics and Statistics* 51:239–246.

Sangvinatsos, A., and J. A. Wachter. 2005. Does the Failure of the Expectations Hypothesis Matter for Long-Term Investors? *Journal of Finance* 60:179–230.

Schlafmann, K., O. Setty, and R. Vestman. 2024. Designing Pension Plans According to Consumption-Savings Theory. Forthcoming in *The Review of Financial Studies*.

Schroder, M., and C. Skiadas. 2002. An Isomorphism Between Asset Pricing Models With and Without Linear Habit Formation. *The Review of Financial Studies* 15:1189–1221.

Shumway, T. 1997. Explaining Returns with Loss Aversion. Working Paper.

The World Bank Group. 2021. The Changing Wealth of Nations 2021. Managing Assets for the Future. Tech. rep., The World Bank Group.

Tversky, A., and D. Kahneman. 1992. Advances in Prospect Theory: Cumulative Representation of Uncertainty. *Journal of Risk and Uncertainty* 5:297–323.

Van Bilsen, S., A. L. Bovenberg, and R. J. A. Laeven. 2020a. Consumption and Portfolio Choice under Internal Multiplicative Habit Formation. *Journal of Financial and Quantitative Analysis* 55:2334–2371.

Van Bilsen, S., R. J. A. Laeven, and Th. E. Nijman. 2020b. Consumption and Portfolio Choice under Loss Aversion and Endogenous Updating of the Reference Level. *Management Science* 66:3799–4358.

Vestman, R. 2019. Limited Stock Market Participation Among Renters and Homeowners. *The Review of Financial Studies* 32:1494–1535.

Viceira, L. M. 2001. Optimal Portfolio Choice for Long-Horizon Investors with Nontradable Labor Income. *Journal of Finance* 56:433–470.

Vissing-Jørgensen, A. 2002. Limited Asset Market Participation and the Elasticity of Intertemporal Substitution. *Journal of Political Economy* 110:825–853.

Wachter, J. A. 2002. Portfolio and Consumption Decisions under Mean-Reverting Returns: An Exact Solution for Complete Markets. *Journal of Financial and Quantitative Analysis* 37:63–91.

Wang, C., N. Wang, and J. Yang. 2016. Optimal Consumption and Savings with Stochastic Income and Recursive Utility. *Journal of Economic Theory* 165:292–331.

Wang, Y., J. Liu, and T. K. Siu. 2024. Investment–Consumption–Insurance Optimisation Problem with Multiple Habit Formation and Non-exponential Discounting. *Finance and Stochastics* 28:161–214.

Yogo, M. 2008. Asset Prices Under Habit Formation and Reference-Dependent Preferences. *Journal of Business and Economic Statistics* 26:539–558.

Internet Appendix

Introduction

The dual portfolio strategy and the dual factor loadings are determined such that the changes in the value of future dual consumption match the changes in dual total wealth. This internet appendix illustrates how we implement this numerically. We need to determine the dual portfolio strategy $\widehat{\omega}(t)$, the dual market price of aggregate consumption risk $\widehat{\lambda}_{\bar{C}}(t)$ and the dual market price of labor income risk $\widehat{\lambda}_Y(t)$. Note that the dual market price of stock return risk $\widehat{\lambda}_S$ is exogenously given (i.e., $\widehat{\lambda}_S = \lambda_S = (\mu_S - r)/\sigma_S$).

Let us denote by T_R the number of working periods (e.g., $T_R = 40$) and by T_D the total number of periods (e.g., $T_D = 60$). We assume that the individual receives dual labor income at the beginning of a period. Dual consumption also takes places at the beginning of a period.

Algorithm

We now describe the *backward induction* algorithm to compute $\widehat{\omega}^*(t)$, $\widehat{\lambda}^*_{\bar{C}}(t)$ and $\widehat{\lambda}^*_Y(t)$ for every $t \in \{1, \ldots, T_D - 1\}$. First, using forward simulation, we construct a grid of $\widehat{Y}_1(t) \equiv Y(t)\,(1 - \beta_Y f_Y(t))\,/(1+\beta_c f_c(t))$, $\widehat{Y}_2(t) \equiv -\beta_{\bar{C}} f_{\bar{C}}(t)\bar{C}(t)/(1+\beta_c f_c(t))$ and $\widehat{m}(t)$.[28] Although $\widehat{m}(t)$ is determined endogenously, we can still construct a grid of $\widehat{m}(t)$.[29] We denote by $\widehat{Y}_1^i(t)$, $\widehat{Y}_2^j(t)$ and $\widehat{m}^k(t)$ the ith grid point of $\widehat{Y}_1(t)$, the jth grid point of $\widehat{Y}_2(t)$ and the kth grid point of $\widehat{m}(t)$, respectively. The corresponding optimal dual portfolio strategy and optimal dual market prices of risk are denoted by $\widehat{\omega}^{*i,j,k}(t)$, $\widehat{\lambda}^{*i,j,k}_{\bar{C}}(t)$ and

[28] Note that $\widehat{Y}(t) = \widehat{Y}_1(t) + \widehat{Y}_2(t)$.

[29] We should define the smallest and largest grid point such that most realizations of $\widehat{m}(t)$ lie between these lower and upper bounds.

$\widehat{\lambda}_Y^{*i,j,k}(t)$. In the remainder, we use bold face to indicate that a value of a variable is *not* known given current information $\{\widehat{Y}_1^i(t), \widehat{Y}_2^j(t), \widehat{m}^k(t)\}$.

Last Period of the Retirement Phase

During retirement, in the numerical examples, labor income is zero. Hence, $\widehat{Y}_1(t) = 0$ for all $t \in \{T_R + 1, \ldots, T_D - 1\}$. In the retirement phase, current information is thus represented by $\left\{\widehat{Y}_2^j(t), \widehat{m}^k(t)\right\}$.

For every combination $\left\{\widehat{Y}_2^j(T_D - 1), \widehat{m}^k(T_D - 1)\right\}$, we determine $\widehat{\omega}^{*j,k}(T_D - 1)$, $\widehat{\lambda}_{\bar{C}}^{*j,k}(T_D - 1)$ and $\widehat{\lambda}_Y^{*j,k}(T_D - 1)$. To do so, we need to derive the change in dual human wealth, the change in the value of future dual consumption and the change in dual financial wealth. We start by deriving the change in dual human wealth.

We find that $\widehat{H}^{j,k}(T_D - 1) = \widehat{Y}_2^j(T_D - 1) + \mathbb{E}_{T_D-1}\left[\frac{\widehat{\boldsymbol{m}}(T_D)}{\widehat{m}^k(T_D-1)}\widehat{\boldsymbol{Y}}_2(T_D)\right]$ and $\widehat{\boldsymbol{H}}(T_D) = \widehat{\boldsymbol{Y}}_2(T_D)$. Hence,

$$\begin{aligned}\widehat{H}^{j,k}(T_D - 1) &= \widehat{Y}_2^j(T_D - 1) + \widehat{Y}_2^j(T_D - 1)\mathbb{E}_{T_D-1}\left[\frac{\widehat{\boldsymbol{m}}(T_D)}{\widehat{m}^k(T_D - 1)}\frac{\widehat{\boldsymbol{Y}}_{\boldsymbol{2}}(\boldsymbol{T_D})}{\widehat{Y}_2^j(T_D - 1)}\right] \\ &= \widehat{Y}_2^j(T_D - 1) + \widehat{Y}_2^j(T_D - 1)e^{-\widehat{r}(T_D-1)+\mu_{\widehat{Y}_2}(T_D-1)+\frac{1}{2}\sigma_{\bar{C}}^2-\sigma_{\bar{C}}\widehat{\lambda}_{\bar{C}}^{j,k}(T_D-1)},\end{aligned} \tag{IA1}$$

with $\mu_{\widehat{Y}_2}(t) \equiv \mu_{\bar{C}} + \mathrm{d}\log f_{\bar{C}}(t)/\mathrm{d}t - \mathrm{d}\log\left(1 + \beta_c f_c(t)\right)/\mathrm{d}t$. Furthermore, we find that

$$\widehat{\boldsymbol{H}}(T_D) = \widehat{Y}_2^j(T_D - 1)e^{\mu_{\widehat{Y}_2}(T_D-1)+\sigma_{\bar{C}}\epsilon_{\bar{C}}(T_D)}. \tag{IA2}$$

Here, $\epsilon_{\bar{C}}(T_D)$ is the unexpected aggregate consumption shock between the beginning of period $T_D - 1$ and the beginning of period T_D. Hence, dual human wealth changes as follows:

$$\begin{aligned}\Delta\widehat{\boldsymbol{H}}(T_D) &= \widehat{\boldsymbol{H}}(T_D) - \widehat{H}^{j,k}(T_D-1)\\ &= \widehat{Y}_2^j(T_D-1)\Big\{e^{\mu_{\widehat{Y}_2}(T_D-1)+\sigma_{\bar{C}}\epsilon_{\bar{C}}(T_D)}\\ &\qquad -e^{-\widehat{r}(T_D-1)+\mu_{\widehat{Y}_2}(T_D-1)+\frac{1}{2}\sigma_{\bar{C}}^2-\sigma_{\bar{C}}\widehat{\lambda}_{\bar{C}}^{j,k}(T_D-1)}-1\Big\}\\ &\approx \widehat{Y}_2^j(T_D-1)\Big\{\widehat{r}(T_D-1)+\sigma_{\bar{C}}\widehat{\lambda}_{\bar{C}}^{j,k}(T_D-1)+\sigma_{\bar{C}}\epsilon_{\bar{C}}(T_D)-1\Big\}.\end{aligned} \tag{IA3}$$

Here, we have used the approximations $e^x \approx 1+x$ and $e^{-\frac{1}{2}\sigma^2+\sigma\epsilon} \approx 1+\sigma\epsilon$ with $\epsilon \sim N(0,1)$.

We now derive the change in the value of future dual consumption. We find

$$\widehat{\boldsymbol{V}}_G(T_D) = \widehat{\boldsymbol{c}}_G(T_D)\mathbb{1}_{\left[\frac{\widehat{\boldsymbol{m}}(T_D)}{\widehat{m}(1)}\le\xi(T_D)\right]}, \tag{IA4}$$

$$\begin{aligned}\widehat{V}_G^{j,k}(T_D-1) &= \widehat{c}_G^k(T_D-1)\mathbb{1}_{\left[\frac{\widehat{\boldsymbol{m}}(T_D-1)}{\widehat{m}(1)}\le\xi(T_D-1)\right]}\\ &\quad+\mathbb{E}_{T_D-1}\left[\frac{\widehat{\boldsymbol{m}}(T_D)}{\widehat{m}^k(T_D-1)}\widehat{\boldsymbol{c}}_G(T_D)\mathbb{1}_{\left[\frac{\widehat{\boldsymbol{m}}(T_D)}{\widehat{m}(1)}\le\xi(T_D)\right]}\right],\end{aligned} \tag{IA5}$$

and

$$\widehat{\boldsymbol{V}}_L(T_D) = \widehat{\boldsymbol{c}}_L(T_D)\mathbb{1}_{\left[\frac{\widehat{\boldsymbol{m}}(T_D)}{\widehat{m}(1)}>\xi(T_D)\right]}, \tag{IA6}$$

$$\begin{aligned}\widehat{V}_L^{j,k}(T_D-1) &= \widehat{c}_L^k(T_D-1)\mathbb{1}_{\left[\frac{\widehat{\boldsymbol{m}}(T_D-1)}{\widehat{m}(1)}>\xi(T_D-1)\right]}\\ &\quad+\mathbb{E}_{T_D-1}\left[\frac{\widehat{\boldsymbol{m}}(T_D)}{\widehat{m}^k(T_D-1)}\widehat{\boldsymbol{c}}_L(T_D)\mathbb{1}_{\left[\frac{\widehat{\boldsymbol{m}}(T_D)}{\widehat{m}(1)}>\xi(T_D)\right]}\right].\end{aligned} \tag{IA7}$$

In what follows we assume $0 < \gamma_L \le 1$ and $L(t) = \infty$ for every t. Under these assumptions,

$$\widehat{c}_G^k(T_D-1) = \left(e^{\delta(T_D-2)}y\frac{\widehat{m}^k(T_D-1)}{\gamma_G\widehat{m}(1)}\right)^{-\frac{1}{1-\gamma_G}}, \tag{IA8}$$

$$\widehat{c}_L^k(T_D-1) = -\left(e^{\delta(T_D-2)}y\frac{\widehat{m}^k(T_D-1)}{\eta\gamma_L\widehat{m}(1)}\right)^{-\frac{1}{1-\gamma_L}}. \tag{IA9}$$

Define $\Gamma(t) \equiv \widehat{\phi}^{\top j,k}(t)\rho\widehat{\phi}^{j,k}(t)$. We find:

$$\widehat{\boldsymbol{V}}_G(T_D) = \widehat{c}_G^k(T_D-1)e^{-\frac{\delta}{1-\gamma_G}+\frac{1}{1-\gamma_G}\left(\widehat{r}(T_D-1)+\frac{1}{2}\Gamma(T_D-1)\right)} \times \mathbb{1}_{\left[\frac{\widehat{\boldsymbol{m}}(T_D)}{\widehat{m}(1)}\leq\xi(T_D)\right]}$$
$$\times\, e^{-\frac{1}{1-\gamma_G}\left(\widehat{\phi}_S^{j,k}(T_D-1)\epsilon_S(T_D)+\widehat{\phi}_{\bar{C}}^{j,k}(T_D-1)\epsilon_{\bar{C}}(T_D)+\widehat{\phi}_Y^{j,k}(T_D-1)\epsilon_Y(T_D)\right)}, \tag{IA10}$$

and

$$\begin{aligned}\widehat{V}_G^{j,k}(T_D-1) &= \widehat{c}_G^k(T_D-1)\Bigg(\mathbb{1}_{\left[\frac{\widehat{\boldsymbol{m}}(T_D-1)}{\widehat{m}(1)}\leq\xi(T_D-1)\right]}+\\ &\quad\times\mathbb{E}_{T_D-1}\left[e^{-\frac{\delta}{1-\gamma_G}}\left(\frac{\widehat{\boldsymbol{m}}(T_D)}{\widehat{m}^k(T_D-1)}\right)^{1-\frac{1}{1-\gamma_G}}\mathbb{1}_{\left[\frac{\widehat{\boldsymbol{m}}(T_D)}{\widehat{m}(1)}\leq\xi(T_D)\right]}\right]\Bigg)\\ &= \widehat{c}_G^k(T_D-1)\Bigg\{e^{-\frac{\delta}{1-\gamma_G}-\frac{-\gamma_G}{1-\gamma_G}\left(\widehat{r}(T_D-1)+\frac{1}{2}\Gamma(T_D-1)\right)}\\ &\qquad\times e^{\frac{1}{2}\left(\frac{-\gamma_G}{1-\gamma_G}\right)^2\Gamma(T_D-1)}\mathcal{N}\left[d_1\left(\xi(T_D)\right)\right]\Bigg\}\\ &\quad+\widehat{c}_G^k(T_D-1)\mathbb{1}_{\left[\frac{\widehat{\boldsymbol{m}}(T_D-1)}{\widehat{m}(1)}\leq\xi(T_D-1)\right]},\end{aligned} \tag{IA11}$$

with $\mathcal{N}$ denoting the cumulative distribution function of a standard normal random variable and

$$\begin{aligned}d_1(x) &= \frac{1}{\sqrt{\Gamma(T_D-1)}}\cdot\left[\log(x)-\log\left(\frac{\widehat{m}^k(T_D-1)}{\widehat{m}(1)}\right)+\widehat{r}(T_D-1)-\frac{1}{2}\Gamma(T_D-1)\right]\\ &\quad+\frac{\sqrt{\Gamma(T_D-1)}}{1-\gamma_G}.\end{aligned} \tag{IA12}$$

Similarly,

$$\widehat{\boldsymbol{V}}_L(T_D) = -\widehat{c}_L^k(T_D-1)e^{-\frac{\delta}{1-\gamma_L}+\frac{1}{1-\gamma_L}\left(\widehat{r}(T_D-1)+\frac{1}{2}\Gamma(T_D-1)\right)} \times \mathbb{1}_{\left[\frac{\widehat{\boldsymbol{m}}(T_D)}{\widehat{m}(1)}>\xi(T_D)\right]}$$
$$\times\, e^{-\frac{1}{1-\gamma_L}\left(\widehat{\phi}_S^{j,k}(T_D-1)\epsilon_S(T_D)+\widehat{\phi}_{\bar{C}}^{j,k}(T_D-1)\epsilon_{\bar{C}}(T_D)+\widehat{\phi}_Y^{j,k}(T_D-1)\epsilon_Y(T_D)\right)}, \tag{IA13}$$

and

$$
\begin{aligned}
\widehat{V}_L^{j,k}(T_D-1) &= -\widehat{c}_L^{\,k}(T_D-1)\Bigg(\mathbb{1}_{\left[\frac{\widehat{\boldsymbol{m}}(T_D-1)}{\widehat{m}(1)}>\xi(T_D-1)\right]}+ \\
&\qquad \times\mathbb{E}_{T_D-1}\left[e^{-\frac{\delta}{1-\gamma_L}}\left(\frac{\widehat{\boldsymbol{m}}(T_D)}{\widehat{m}^k(T_D-1)}\right)^{1-\frac{1}{1-\gamma_L}}\mathbb{1}_{\left[\frac{\widehat{\boldsymbol{m}}(T_D)}{\widehat{m}(1)}>\xi(T_D)\right]}\right]\Bigg) \\
&= \widehat{c}_L^{\,k}(T_D-1)\Bigg\{e^{-\frac{\delta}{1-\gamma_L}-\frac{-\gamma_L}{1-\gamma_L}\left(\widehat{r}(T_D-1)+\frac{1}{2}\Gamma(T_D-1)\right)} \\
&\qquad\qquad \times e^{\frac{1}{2}\left(\frac{-\gamma_L}{1-\gamma_L}\right)^2\Gamma(T_D-1)}\left(1-\mathcal{N}\left[d_2\left(\xi(T_D)\right)\right]\right)\Bigg\} \\
&\quad + \widehat{c}_L^{\,k}(T_D-1)\mathbb{1}_{\left[\frac{\widehat{\boldsymbol{m}}(T_D-1)}{\widehat{m}(1)}>\xi(T_D-1)\right]},
\end{aligned} \tag{IA14}
$$

with

$$
\begin{aligned}
d_2(x) &= \frac{1}{\sqrt{\Gamma(T_D-1)}}\cdot\left[\log(x)-\log\left(\frac{\widehat{m}^k(T_D-1)}{\widehat{m}(1)}\right)+\widehat{r}(T_D-1)-\frac{1}{2}\Gamma(T_D-1)\right] \\
&\quad + \frac{\sqrt{\Gamma(T_D-1)}}{1-\gamma_L}.
\end{aligned} \tag{IA15}
$$

The change in the value of future dual consumption is thus given by

$$
\Delta\widehat{\boldsymbol{V}}(T_D) = \widehat{\boldsymbol{V}}_G(T_D) - \widehat{V}_G^{j,k}(T_D-1) + \widehat{\boldsymbol{V}}_L(T_D) - \widehat{V}_L^{j,k}(T_D-1). \tag{IA16}
$$

We finally derive the change in dual financial wealth. We find

$$
\begin{aligned}
\widehat{\boldsymbol{F}}(T_D) &= \widehat{Y}_2^j(T_D-1) - \widehat{c}^{\,k}(T_D-1) \\
&\qquad + \widehat{F}^{j,k}(T_D-1)e^{\widehat{r}(T_D-1)+\widehat{\omega}^{j,k}(T_D-1)\lambda_S\sigma_S-\frac{1}{2}\widehat{\omega}^{j,k}(T_D-1)^2\sigma_S^2+\widehat{\omega}^{j,k}(T_D-1)\sigma_S\epsilon_S(T_D)},
\end{aligned} \tag{IA17}
$$

with

$$
\widehat{F}^{j,k}(T_D-1) = \widehat{c}^{\,k}(T_D-1) - \widehat{Y}_2^j(T_D-1). \tag{IA18}
$$

Hence,

$$
\begin{aligned}
\Delta\widehat{\boldsymbol{F}}(T_D) &= \widehat{\boldsymbol{F}}(T_D) - \widehat{F}^{j,k}(T_D-1) \\
&\approx \widehat{Y}_2^j(T_D-1) - \widehat{c}^{\,k}(T_D-1) \\
&\quad + \widehat{F}^{j,k}(T_D-1)\left(\widehat{r}(T_D-1) + \widehat{\omega}^{j,k}(T_D-1)\lambda_S\sigma_S\right. \\
&\qquad\qquad \left. + \widehat{\omega}^{j,k}(T_D-1)\sigma_S\epsilon_S(T_D)\right).
\end{aligned} \tag{IA19}
$$

Using $\Delta\widehat{\boldsymbol{V}}(T_D) = \Delta\widehat{\boldsymbol{H}}(T_D) + \Delta\widehat{\boldsymbol{F}}(T_D)$, we numerically determine $\widehat{\omega}^{*j,k}(T_D-1)$ and $\widehat{\phi}^{*j,k}_{\bar{C}}(T_D-1)$. Since labor income is zero during retirement, we have $\widehat{\phi}^{*j,k}_{Y}(T_D-1) = 0$. Note that for all t,

$$
\widehat{\phi}^{j,k}_S(t) = -\widehat{\lambda}_S - \rho_{1,2}^{-1}\widehat{\lambda}^{j,k}_{\bar{C}}(t) - \rho_{1,3}^{-1}\widehat{\lambda}^{j,k}_Y(t), \tag{IA20}
$$

$$
\widehat{\lambda}^{j,k}_{\bar{C}}(t) = -\rho_{2,1}\widehat{\phi}^{j,k}_S(t) - \widehat{\phi}^{j,k}_{\bar{C}}(t) - \rho_{2,3}\widehat{\phi}^{j,k}_Y(t), \tag{IA21}
$$

$$
\widehat{\lambda}^{j,k}_Y(t) = -\rho_{3,1}\widehat{\phi}^{j,k}_S(t) - \rho_{3,2}\widehat{\phi}^{j,k}_{\bar{C}}(t) - \widehat{\phi}^{j,k}_Y(t). \tag{IA22}
$$

Hence,

$$
\begin{aligned}
\widehat{\phi}^{j,k}_S(T_D-1) &= \frac{-\widehat{\lambda}_S + \rho_{1,2}^{-1}\left(\widehat{\phi}^{j,k}_{\bar{C}}(T_D-1) + \rho_{2,3}\widehat{\phi}^{j,k}_Y(T_D-1)\right)}{1-\rho_{1,2}^{-1}\rho_{2,1} - \rho_{1,3}^{-1}\rho_{3,1}} \\
&\quad + \frac{\rho_{1,3}^{-1}\left(\widehat{\phi}^{j,k}_Y(T_D-1) + \rho_{3,2}\widehat{\phi}^{j,k}_{\bar{C}}(T_D-1)\right)}{1-\rho_{1,2}^{-1}\rho_{2,1} - \rho_{1,3}^{-1}\rho_{3,1}}.
\end{aligned} \tag{IA23}
$$

Remaining Periods of the Retirement Phase

We now determine $\widehat{\omega}^{*j,k}(t)$, $\widehat{\lambda}_{\bar{C}}^{*j,k}(t)$ and $\widehat{\lambda}_{Y}^{*j,k}(t)$ for all $t \in \{T_R, \ldots, T_D - 2\}$. We start by deriving the change in dual human wealth. We find

$$\widehat{\boldsymbol{H}}(t+1) = \mathbb{E}_{t+1}\left[\sum_{l=0}^{T_D-t-1} \frac{\widehat{\boldsymbol{m}}(t+1+l)}{\widehat{\boldsymbol{m}}(t+1)} \widehat{\boldsymbol{Y}}_2(t+1+l)\right], \tag{IA24}$$

$$\begin{aligned}\widehat{H}^{j,k}(t) &= \widehat{Y}_2^j(t) + \mathbb{E}_t\left[\sum_{l=0}^{T_D-t-1} \frac{\widehat{\boldsymbol{m}}(t+l+1)}{\widehat{m}^k(t)} \widehat{\boldsymbol{Y}}_2(t+l+1)\right] \\ &= \widehat{Y}_2^j(t) + \mathbb{E}_t\left\{\frac{\widehat{\boldsymbol{m}}(t+1)}{\widehat{m}^k(t)} \mathbb{E}_{t+1}\left[\sum_{l=0}^{T_D-t-1} \frac{\widehat{\boldsymbol{m}}(t+l+1)}{\widehat{\boldsymbol{m}}(t+1)} \widehat{\boldsymbol{Y}}_2(t+l+1)\right]\right\} \\ &= \widehat{Y}_2^j(t) + \mathbb{E}_t\left[\frac{\widehat{\boldsymbol{m}}(t+1)}{\widehat{m}^k(t)} \widehat{\boldsymbol{H}}(t+1)\right].\end{aligned} \tag{IA25}$$

In the previous step, we have determined $\widehat{\boldsymbol{H}}(t+1)$. Using OLS regression, we now determine the coefficients $\eta_0^H(t+1)$, $\eta_1^H(t+1)$ and $\eta_2^H(t+1)$ such that

$$\log\left(-\widehat{\boldsymbol{H}}(t+1)\right) \approx \eta_0^H(t+1) + \eta_1^H(t+1)\log\left(-\widehat{\boldsymbol{Y}}_2(t+1)\right) + \eta_2^H(t+1)\log\widehat{\boldsymbol{m}}(t+1). \tag{IA26}$$

Hence,

$$\begin{aligned}\widehat{H}^{j,k}(t) = \widehat{Y}_2^j(t) - e^{\eta_0^H(t+1)}\left(-\widehat{Y}_2^j(t)\right)^{\eta_1^H(t+1)}\left(\widehat{m}^k(t)\right)^{\eta_2^H(t+1)} \\ \times \mathbb{E}_t\left[\left(\frac{\widehat{\boldsymbol{Y}}_2(t+1)}{\widehat{Y}_2^j(t)}\right)^{\eta_1^H(t+1)}\left(\frac{\widehat{\boldsymbol{m}}(t+1)}{\widehat{m}^k(t)}\right)^{\eta_2^H(t+1)+1}\right].\end{aligned} \tag{IA27}$$

We find that

$$\begin{aligned}\Delta\widehat{\boldsymbol{H}}(t+1) \approx -e^{\eta_0^H(t+1)}\left(-\widehat{Y}_2^j(t)\right)^{\eta_1^H(t+1)}\left(\widehat{m}^k(t)\right)^{\eta_2^H(t+1)} \\ \times\left\{\ldots + \left(\eta_1^H(t+1)\sigma_{\bar{C}} + \eta_2^H(t+1)\widehat{\phi}_{\bar{C}}^{j,k}(t)\right)\epsilon_{\bar{C}}(t+1)\right. \\ \left. + \eta_2^H(t+1)\widehat{\phi}_S^{j,k}(t)\epsilon_S(t+1)\right\} - \widehat{Y}_2^j(t).\end{aligned} \tag{IA28}$$

We now derive the change in the value of future dual consumption. We find

$$\widehat{\boldsymbol{V}}_G(t+1) = \mathbb{E}_{t+1}\left[\sum_{l=0}^{T_D-t-1} \frac{\widehat{\boldsymbol{m}}(t+1+l)}{\widehat{\boldsymbol{m}}(t+1)} \widehat{\boldsymbol{c}}_G(t+1+l) \mathbb{1}_{\left[\frac{\widehat{\boldsymbol{m}}(t+1+l)}{\widehat{m}(0)} \leq \xi(t+1+l)\right]}\right] \tag{IA29}$$

$$\begin{aligned}
\widehat{V}_G^{j,k}(t) &= \widehat{c}_G^k(t) \mathbb{1}_{\left[\frac{\widehat{m}(t)}{\widehat{m}(0)} \leq \xi(t)\right]} + \mathbb{E}_t\left[\sum_{l=0}^{T_D-t-1} \frac{\widehat{\boldsymbol{m}}(t+1+l)}{\widehat{m}^k(t)} \widehat{\boldsymbol{c}}_G(t+1+l)\right. \\
&\qquad \left. \mathbb{1}_{\left[\frac{\widehat{\boldsymbol{m}}(t+1+l)}{\widehat{m}(0)} \leq \xi(t+1+l)\right]}\right] \\
&= \widehat{c}_G^k(t) \mathbb{1}_{\left[\frac{\widehat{m}(t)}{\widehat{m}(0)} \leq \xi(t)\right]} + \mathbb{E}_t\left\{\frac{\widehat{\boldsymbol{m}}(t+1)}{\widehat{m}^k(t)} \mathbb{E}_{t+1}\left[\sum_{l=0}^{T_D-t-1} \frac{\widehat{\boldsymbol{m}}(t+1+l)}{\widehat{\boldsymbol{m}}(t+1)}\right.\right. \\
&\qquad \left.\left. \widehat{\boldsymbol{c}}_G(t+l+1) \mathbb{1}_{\left[\frac{\widehat{\boldsymbol{m}}(t+1+l)}{\widehat{m}(0)} \leq \xi(t+1+l)\right]}\right]\right\} \\
&= \widehat{c}_G^k(t) \mathbb{1}_{\left[\frac{\widehat{m}(t)}{\widehat{m}(0)} \leq \xi(t)\right]} + \mathbb{E}_t\left[\frac{\widehat{\boldsymbol{m}}(t+1)}{\widehat{m}^k(t)} \widehat{\boldsymbol{V}}_G(t+1)\right],
\end{aligned} \tag{IA30}$$

and

$$\widehat{\boldsymbol{V}}_L(t+1) = \mathbb{E}_{t+1}\left[\sum_{l=0}^{T_D-t-1} \frac{\widehat{\boldsymbol{m}}(t+1+l)}{\widehat{\boldsymbol{m}}(t+1)} \widehat{\boldsymbol{c}}_L(t+1+l) \mathbb{1}_{\left[\frac{\widehat{\boldsymbol{m}}(t+1+l)}{\widehat{m}(0)} > \xi(t+1+l)\right]}\right], \tag{IA31}$$

$$\begin{aligned}
\widehat{V}_L^{j,k}(t) &= \widehat{c}_L^k(t) \mathbb{1}_{\left[\frac{\widehat{m}(t)}{\widehat{m}(0)} > \xi(t)\right]} + \mathbb{E}_t\left[\sum_{l=0}^{T_D-t-1} \frac{\widehat{\boldsymbol{m}}(t+1+l)}{\widehat{m}^k(t)} \widehat{\boldsymbol{c}}_L(t+1+l)\right. \\
&\qquad \left. \mathbb{1}_{\left[\frac{\widehat{\boldsymbol{m}}(t+1+l)}{\widehat{m}(0)} > \xi(t+1+l)\right]}\right] \\
&= \widehat{c}_L^k(t) \mathbb{1}_{\left[\frac{\widehat{m}(t)}{\widehat{m}(0)} > \xi(t)\right]} + \mathbb{E}_t\left\{\frac{\widehat{\boldsymbol{m}}(t+1)}{\widehat{m}^k(t)} \mathbb{E}_{t+1}\left[\sum_{l=0}^{T_D-t-1} \frac{\widehat{\boldsymbol{m}}(t+1+l)}{\widehat{\boldsymbol{m}}(t+1)}\right.\right. \\
&\qquad \left.\left. \widehat{\boldsymbol{c}}_L(t+l+1) \mathbb{1}_{\left[\frac{\widehat{\boldsymbol{m}}(t+1+l)}{\widehat{m}(0)} > \xi(t+1+l)\right]}\right]\right\} \\
&= \widehat{c}_L^k(t) \mathbb{1}_{\left[\frac{\widehat{m}(t)}{\widehat{m}(0)} > \xi(t)\right]} + \mathbb{E}_t\left[\frac{\widehat{\boldsymbol{m}}(t+1)}{\widehat{m}^k(t)} \widehat{\boldsymbol{V}}_L(t+1)\right],
\end{aligned} \tag{IA32}$$

with

$$\widehat{c}_G^k(t) = \left(e^{\delta(t-1)} y \frac{\widehat{m}^k(t)}{\gamma_G \widehat{m}(1)}\right)^{-\frac{1}{1-\gamma_G}}, \tag{IA33}$$

$$\widehat{c}_L^k(t) = -\left(e^{\delta(t-1)} y \frac{\widehat{m}^k(t)}{\eta\gamma_L \widehat{m}(1)}\right)^{-\frac{1}{1-\gamma_L}}. \tag{IA34}$$

In the previous step, we have determined $\widehat{\boldsymbol{V}}_G(t+1)$ and $\widehat{\boldsymbol{V}}_L(t+1)$. Using OLS regressions, we now determine the coefficients $\eta_{0,G}^V(t+1)$, $\eta_{1,G}^V(t+1)$, $\eta_{2,G}^V(t+1)$, $\eta_{0,L}^V(t+1)$, $\eta_{1,l}^V(t+1)$ and $\eta_{2,l}^V(t+1)$ such that

$$\begin{aligned}
\log \widehat{\boldsymbol{V}}_G(t+1) &\approx \eta_{0,G}^V(t+1) + \eta_{1,G}^V(t+1)\log\left(-\widehat{\boldsymbol{Y}}_2(t+1)\right) \\
&\quad + \eta_{2,G}^V(t+1)\log\widehat{\boldsymbol{m}}(t+1), \\
\log\left(-\widehat{\boldsymbol{V}}_L(t+1)\right) &\approx \eta_{0,L}^V(t+1) + \eta_{1,L}^V(t+1)\log\left(-\widehat{\boldsymbol{Y}}_{2,L}(t+1)\right) \\
&\quad + \eta_{2,L}^V(t+1)\log\widehat{\boldsymbol{m}}(t+1).
\end{aligned} \tag{IA35}$$

Hence,

$$\begin{aligned}
\widehat{V}^{j,k}(t) &= \widehat{V}_G^{j,k}(t) + \widehat{V}_L^{j,k}(t) \\
&= \widehat{c}^{\,k}(t) + e^{\eta_{0,G}^V(t+1)}\left(-\widehat{Y}_2^j(t)\right)^{\eta_{1,G}^V(t+1)}\left(\widehat{m}^k(t)\right)^{\eta_{2,G}^V(t+1)} \\
&\quad \times \mathbb{E}_t\left[\left(\frac{\widehat{\boldsymbol{Y}}_2(t+1)}{\widehat{Y}_2^j(t)}\right)^{\eta_{1,G}^V(t+1)}\left(\frac{\widehat{\boldsymbol{m}}(t+1)}{\widehat{m}^k(t)}\right)^{\eta_{2,G}^V(t+1)+1}\right] \\
&\quad - e^{\eta_{0,L}^V(t+1)}\left(-\widehat{Y}_2^j(t)\right)^{\eta_{1,L}^V(t+1)}\left(\widehat{m}^k(t)\right)^{\eta_{2,L}^V(t+1)} \\
&\quad \times \mathbb{E}_t\left[\left(\frac{\widehat{\boldsymbol{Y}}_2(t+1)}{\widehat{Y}_2^j(t)}\right)^{\eta_{1,L}^V(t+1)}\left(\frac{\widehat{\boldsymbol{m}}(t+1)}{\widehat{m}^k(t)}\right)^{\eta_{2,L}^V(t+1)+1}\right].
\end{aligned} \tag{IA36}$$

We now find that

$$
\begin{aligned}
\Delta \widehat{\boldsymbol{V}}(t+1) \approx\ & e^{\eta^V_{0,G}(t+1)} \left(-\widehat{Y}^j_2(t)\right)^{\eta^V_{1,G}(t+1)} \left(\widehat{m}^k(t)\right)^{\eta^V_{2,G}(t+1)} \\
& \times \Big\{ \ldots + \left(\eta^V_{1,G}(t+1)\sigma_{\bar{C}} + \eta^V_{2,G}(t+1)\widehat{\phi}^{j,k}_{\bar{C}}(t)\right) \epsilon_{\bar{C}}(t+1) \\
& \quad + \eta^V_{2,G}(t+1)\widehat{\phi}^{j,k}_S(t)\epsilon_S(t+1) \Big\} \\
& - e^{\eta^V_{0,L}(t+1)} \left(-\widehat{Y}^j_2(t)\right)^{\eta^V_{1,L}(t+1)} \left(\widehat{m}^k(t)\right)^{\eta^V_{2,L}(t+1)} \\
& \times \Big\{ \ldots + \left(\eta^V_{1,L}(t+1)\sigma_{\bar{C}} + \eta^V_{2,L}(t+1)\widehat{\phi}^{j,k}_{\bar{C}}(t)\right) \epsilon_{\bar{C}}(t+1) \\
& \quad + \eta^V_{2,L}(t+1)\widehat{\phi}^{j,k}_S(t)\epsilon_S(t+1) \Big\} - \widehat{c}^{\,k}(t).
\end{aligned}
\tag{IA37}
$$

We finally derive the change in dual financial wealth. We find

$$
\Delta \widehat{\boldsymbol{F}}(t+1) \approx \widehat{Y}^j_2(t) - \widehat{c}^{\,k}(t) + \widehat{F}^{j,k}(t) \left(\widehat{r}(t) + \widehat{\omega}^{j,k}(t)\lambda_S\sigma_S + \widehat{\omega}^{j,k}(t)\sigma_S\epsilon_S(t+1)\right). \tag{IA38}
$$

Here,

$$
\begin{aligned}
\widehat{F}^{j,k}(t) =\ & e^{\eta^V_{0,G}(t+1)} \left(-\widehat{Y}^j_2(t)\right)^{\eta^V_{1,G}(t+1)} \left(\widehat{m}^k(t)\right)^{\eta^V_{2,G}(t+1)} \\
& - e^{\eta^V_{0,L}(t+1)} \left(-\widehat{Y}^j_2(t)\right)^{\eta^V_{1,L}(t+1)} \left(\widehat{m}^k(t)\right)^{\eta^V_{2,L}(t+1)} \\
& + e^{\eta^H_0(t+1)} \left(-\widehat{Y}^j_2(t)\right)^{\eta^H_1(t+1)} \left(\widehat{m}^k(t)\right)^{\eta^H_2(t+1)}.
\end{aligned}
\tag{IA39}
$$

Using the condition $\Delta \widehat{\boldsymbol{V}}(t+1) = \Delta \widehat{\boldsymbol{H}}(t+1) + \Delta \widehat{\boldsymbol{F}}(t+1)$, we arrive at

$$
\begin{aligned}
\widehat{\omega}^{*j,k}(t) =\ & \eta^V_{2,G}(t+1)\widehat{\phi}^{j,k}_S(t)\frac{\widetilde{V}^{j,k}_G(t)}{\sigma_S\widehat{F}^{j,k}(t)} + \eta^V_{2,L}(t+1)\widehat{\phi}^{j,k}_S(t)\frac{\widetilde{V}^{j,k}_L(t)}{\sigma_S\widehat{F}^{j,k}(t)} \\
& - \eta^H_2(t+1)\widehat{\phi}^{j,k}_S(t)\frac{\widetilde{H}^{j,k}(t)}{\sigma_S\widehat{F}^{j,k}(t)},
\end{aligned}
\tag{IA40}
$$

$$
\widehat{\phi}^{*j,k}_{\bar{C}}(t) = \sigma_{\bar{C}}\frac{\eta^V_{1,G}(t+1)\widetilde{V}^{j,k}_G(t) + \eta^V_{1,L}(t+1)\widetilde{V}^{j,k}_L(t) - \eta^H_1(t+1)\widetilde{H}^{j,k}(t)}{\eta^H_2(t+1)\widetilde{H}^{j,k}(t) - \eta^V_{2,G}(t+1)\widetilde{V}^{j,k}_G(t) - \eta^V_{2,L}(t+1)\widetilde{V}^{j,k}_L(t)}, \tag{IA41}
$$

$$
\widehat{\phi}^{*j,k}_Y(t) = 0, \tag{IA42}
$$

with

$$\widetilde{H}^{j,k}(t) \equiv -e^{\eta_0^H(t+1)} \left(-\widehat{Y}_2^j(t)\right)^{\eta_1^H(t+1)} \left(\widehat{m}^k(t)\right)^{\eta_2^H(t+1)}, \tag{IA43}$$

$$\widetilde{V}_G^{j,k}(t) \equiv e^{\eta_{0,G}^V(t+1)} \left(-\widehat{Y}_2^j(t)\right)^{\eta_{1,G}^V(t+1)} \left(\widehat{m}^k(t)\right)^{\eta_{2,G}^V(t+1)}, \tag{IA44}$$

$$\widetilde{V}_L^{j,k}(t) \equiv -e^{\eta_{0,L}^V(t+1)} \left(-\widehat{Y}_2^j(t)\right)^{\eta_{1,L}^V(t+1)} \left(\widehat{m}^k(t)\right)^{\eta_{2,L}^V(t+1)}. \tag{IA45}$$

Last Period of the Working Phase

Now we determine for every combination $\left\{\widehat{Y}_1^i(T_R-1), \widehat{Y}_2^j(T_R-1), \widehat{m}^k(T_R-1)\right\}$, $\widehat{\omega}^{*i,j,k}(T_R-1)$, $\widehat{\lambda}_{\bar{C}}^{*i,j,k}(T_R-1)$ and $\widehat{\lambda}_Y^{*i,j,k}(T_R-1)$. As before, we need to derive the change in dual human wealth, the change in the value of future dual consumption and the change in dual financial wealth. We start by deriving the change in dual human wealth.

We find that $\widehat{H}_1^{i,j,k}(T_R-1) = \widehat{Y}_1^i(T_R-1) + \mathbb{E}_{T_R-1}\left[\frac{\widehat{\boldsymbol{m}}(T_R)}{\widehat{m}^k(T_R-1)}\widehat{\boldsymbol{Y}}_1(T_R)\right]$ and $\widehat{\boldsymbol{H}}_1(T_R) = \widehat{\boldsymbol{Y}}_1(T_R)$. Hence,

$$\begin{aligned}\widehat{H}_1^{i,j,k}(T_R-1) &= \widehat{Y}_1^i(T_R-1) + \widehat{Y}_1^i(T_R-1)\mathbb{E}_{T_R-1}\left[\frac{\widehat{\boldsymbol{m}}(T_R)}{\widehat{m}^k(T_R-1)}\frac{\widehat{\boldsymbol{Y}}_{\boldsymbol{1}}(\boldsymbol{T_R})}{\widehat{Y}_1^i(T_R-1)}\right] \\ &= \widehat{Y}_1^i(T_R-1) + \widehat{Y}_1^i(T_R-1)e^{-\widehat{r}(T_R-1)+\mu_{\widehat{Y}_1}(T_R-1)+\frac{1}{2}\sigma_Y^2-\sigma_Y\widehat{\lambda}_Y^{i,j,k}(T_R-1)},\end{aligned} \tag{IA46}$$

with $\mu_{\widehat{Y}_1}(t) \equiv \mu_Y + \mathrm{d}\log\left(1-\beta_Y f_Y(t)\right)/\mathrm{d}t - \mathrm{d}\log\left(1+\beta_c f_c(t)\right)/\mathrm{d}t$. Furthermore, we find that

$$\widehat{\boldsymbol{H}}_1(T_R) = \widehat{Y}_1^i(T_R-1)e^{\mu_{\widehat{Y}_1}(T_R-1)+\sigma_Y\epsilon_Y(T_R)}. \tag{IA47}$$

Here, $\epsilon_Y(T_R)$ is the unexpected non-tradable labor income shock between the beginning of period T_R-1 and the beginning of period T_R. The first part of dual human wealth changes as follows:

$$\begin{aligned}\Delta\widehat{\boldsymbol{H}}_1(T_R) &= \widehat{\boldsymbol{H}}_1(T_R) - \widehat{H}_1^{i,j,k}(T_R-1)\\ &= \widehat{Y}_1^i(T_R-1)\Big\{e^{\mu_{\widehat{Y}_1}(T_R-1)+\sigma_Y\epsilon_Y(T_R)}\\ &\qquad -e^{-\widehat{r}(T_R-1)+\mu_{\widehat{Y}_1}(T_R-1)+\frac{1}{2}\sigma_Y^2-\sigma_Y\widehat{\lambda}_Y^{i,j,k}(T_R-1)}-1\Big\}\\ &\approx \widehat{Y}_1^i(T_R-1)\Big\{\widehat{r}(T_R-1)+\sigma_Y\widehat{\lambda}_Y^{i,j,k}(T_R-1)+\sigma_Y\epsilon_Y(T_R)-1\Big\}.\end{aligned} \tag{IA48}$$

Here, we have used the approximations $e^x \approx 1+x$ and $e^{-\frac{1}{2}\sigma^2+\sigma\epsilon} \approx 1+\sigma\epsilon$ with $\epsilon \sim N(0,1)$.

We now derive the second part of dual human wealth. We find

$$\widehat{\boldsymbol{H}}_2(T_R) = \mathbb{E}_{T_R}\left[\sum_{l=0}^{T_D-T_R}\frac{\widehat{\boldsymbol{m}}(T_R+l)}{\widehat{\boldsymbol{m}}(T_R)}\widehat{\boldsymbol{Y}}_2(T_R+l)\right], \tag{IA49}$$

$$\begin{aligned}\widehat{H}_2^{i,j,k}(T_R-1) &= \widehat{Y}_2^j(T_R-1) + \mathbb{E}_{T_R-1}\left[\sum_{l=0}^{T_D-T_R}\frac{\widehat{\boldsymbol{m}}(T_R+l)}{\widehat{m}^k(T_R-1)}\widehat{\boldsymbol{Y}}_2(T_R+l)\right]\\ &= \widehat{Y}_2^j(T_R-1) + \mathbb{E}_{T_R-1}\left\{\frac{\widehat{\boldsymbol{m}}(T_R)}{\widehat{m}^k(T_R-1)}\right.\\ &\qquad\left.\mathbb{E}_{T_R}\left[\sum_{l=0}^{T_D-T_R}\frac{\widehat{\boldsymbol{m}}(T_R+l)}{\widehat{\boldsymbol{m}}(T_R)}\widehat{\boldsymbol{Y}}_2(T_R+l)\right]\right\}\\ &= \widehat{Y}_2^j(T_R-1) + \mathbb{E}_{T_R-1}\left[\frac{\widehat{\boldsymbol{m}}(t+1)}{\widehat{m}^k(T_R-1)}\widehat{\boldsymbol{H}}_2(T_R)\right].\end{aligned} \tag{IA50}$$

In the previous step, we have determined $\widehat{\boldsymbol{H}}_2(T_R) = \widehat{\boldsymbol{H}}(T_R)$. Using OLS regression, we now determine the coefficients $\eta_0^{H_2}(T_R)$, $\eta_1^{H_2}(T_R)$ and $\eta_2^{H_2}(T_R)$ such that

$$\log\Big(-\widehat{\boldsymbol{H}}_2(T_R)\Big) \approx \eta_0^{H_2}(T_R) + \eta_1^{H_2}(T_R)\log\Big(-\widehat{\boldsymbol{Y}}_2(T_R)\Big) + \eta_2^{H_2}(T_R)\log\widehat{\boldsymbol{m}}(T_R). \tag{IA51}$$

Hence,

$$\begin{aligned}\widehat{H}_2^{i,j,k}(T_R-1) &= \widehat{Y}_2^j(T_R-1) \\ &\quad - e^{\eta_0^{H_2}(T_R)}\left(-\widehat{Y}_2^j(T_R-1)\right)^{\eta_1^{H_2}(T_R)}\left(\widehat{m}^k(T_R-1)\right)^{\eta_2^{H_2}(T_R)} \\ &\quad \times \mathbb{E}_{T_R-1}\left[\left(\frac{\widehat{\boldsymbol{Y}}_2(T_R)}{\widehat{Y}_2^j(T_R-1)}\right)^{\eta_1^{H_2}(T_R)}\left(\frac{\widehat{\boldsymbol{m}}(T_R)}{\widehat{m}^k(T_R-1)}\right)^{\eta_2^{H_2}(T_R)+1}\right].\end{aligned} \tag{IA52}$$

We find that

$$\begin{aligned}\Delta\widehat{\boldsymbol{H}}_2(T_R) \approx& -e^{\eta_0^{H_2}(T_R)}\left(-\widehat{Y}_2^j(T_R-1)\right)^{\eta_1^{H_2}(T_R)}\left(\widehat{m}^k(T_R-1)\right)^{\eta_2^{H_2}(T_R)} \\ & \times\Big\{\ldots+\left(\eta_1^{H_2}(T_R)\sigma_{\bar{C}}+\eta_2^{H_2}(T_R)\widehat{\phi}_{\bar{C}}^{i,j,k}(T_R-1)\right)\epsilon_{\bar{C}}(T_R) \\ & \quad +\eta_2^{H_2}(T_R)\widehat{\phi}_Y^{i,j,k}(T_R-1)\epsilon_Y(T_R)+\eta_2^{H_2}(T_R)\widehat{\phi}_S^{i,j,k}(T_R-1)\epsilon_S(T_R)\Big\} \\ & -\widehat{Y}_2^j(T_R-1).\end{aligned} \tag{IA53}$$

We now derive the change in the value of future dual consumption. We find

$$\widehat{\boldsymbol{V}}_G(T_R) = \mathbb{E}_{T_R}\left[\sum_{l=0}^{T_D-T_R}\frac{\widehat{\boldsymbol{m}}(T_R+l)}{\widehat{\boldsymbol{m}}(T_R)}\widehat{\boldsymbol{c}}_G(T_R+l)\mathbb{1}_{\left[\frac{\widehat{\boldsymbol{m}}(T_R+l)}{\widehat{m}(0)}\leq\xi(T_R+l)\right]}\right] \tag{IA54}$$

$$\begin{aligned}\widehat{V}_G^{j,k}(T_R-1) &= \widehat{c}_G^k(T_R-1)\mathbb{1}_{\left[\frac{\widehat{m}(T_R-1)}{\widehat{m}(0)}\leq\xi(T_R-1)\right]}+\mathbb{E}_{T_R-1}\left[\sum_{l=0}^{T_D-T_R}\frac{\widehat{\boldsymbol{m}}(T_R+l)}{\widehat{m}^k(T_R-1)}\right. \\ &\qquad\qquad \left.\widehat{\boldsymbol{c}}_G(T_R+l)\mathbb{1}_{\left[\frac{\widehat{\boldsymbol{m}}(T_R+l)}{\widehat{m}(0)}\leq\xi(T_R+l)\right]}\right] \\ &= \widehat{c}_G^k(T_R-1)\mathbb{1}_{\left[\frac{\widehat{m}(T_R-1)}{\widehat{m}(0)}\leq\xi(T_R-1)\right]}+\mathbb{E}_{T_R-1}\left\{\frac{\widehat{\boldsymbol{m}}(T_R)}{\widehat{m}^k(T_R-1)}\right. \\ &\qquad \left.\mathbb{E}_{T_R}\left[\sum_{l=0}^{T_D-T_R}\frac{\widehat{\boldsymbol{m}}(T_R+l)}{\widehat{\boldsymbol{m}}(T_R)}\widehat{\boldsymbol{c}}_G(T_R+l)\mathbb{1}_{\left[\frac{\widehat{\boldsymbol{m}}(T_R+l)}{\widehat{m}(0)}\leq\xi(T_R+l)\right]}\right]\right\} \\ &= \widehat{c}_G^k(T_R-1)\mathbb{1}_{\left[\frac{\widehat{m}(T_R-1)}{\widehat{m}(0)}\leq\xi(T_R-1)\right]} \\ &\qquad +\mathbb{E}_{T_R-1}\left[\frac{\widehat{\boldsymbol{m}}(T_R)}{\widehat{m}^k(T_R-1)}\widehat{\boldsymbol{V}}_G(T_R)\right],\end{aligned} \tag{IA55}$$

and

$$\widehat{\boldsymbol{V}}_L(T_R) = \mathbb{E}_{T_R}\left[\sum_{l=0}^{T_D-T_R} \frac{\widehat{\boldsymbol{m}}(T_R+l)}{\widehat{\boldsymbol{m}}(T_R)} \widehat{\boldsymbol{c}}_L(T_R+l)\mathbb{1}_{\left[\frac{\widehat{\boldsymbol{m}}(T_R+l)}{\widehat{m}(0)}>\xi(T_R+l)\right]}\right], \tag{IA56}$$

$$\begin{aligned}
\widehat{V}_L^{j,k}(T_R-1) &= \widehat{c}_L^k(T_R-1)\mathbb{1}_{\left[\frac{\widehat{m}(T_R-1)}{\widehat{m}(0)}>\xi(T_R-1)\right]} + \mathbb{E}_{T_R-1}\left[\sum_{l=0}^{T_D-T_R} \frac{\widehat{\boldsymbol{m}}(T_R+l)}{\widehat{m}^k(T_R-1)}\right.\\
&\qquad\qquad \left.\widehat{\boldsymbol{c}}_L(T_R+l)\mathbb{1}_{\left[\frac{\widehat{\boldsymbol{m}}(T_R+l)}{\widehat{m}(0)}>\xi(T_R+l)\right]}\right]\\
&= \widehat{c}_L^k(T_R-1)\mathbb{1}_{\left[\frac{\widehat{m}(T_R-1)}{\widehat{m}(0)}>\xi(T_R-1)\right]} + \mathbb{E}_{T_R-1}\left\{\frac{\widehat{\boldsymbol{m}}(T_R)}{\widehat{m}^k(T_R-1)}\right.\\
&\qquad \left.\mathbb{E}_{T_R}\left[\sum_{l=0}^{T_D-T_R} \frac{\widehat{\boldsymbol{m}}(T_R+l)}{\widehat{\boldsymbol{m}}(T_R)} \widehat{\boldsymbol{c}}_L(T_R+l)\mathbb{1}_{\left[\frac{\widehat{\boldsymbol{m}}(T_R+l)}{\widehat{m}(0)}>\xi(T_R+l)\right]}\right]\right\}\\
&= \widehat{c}_L^k(T_R-1)\mathbb{1}_{\left[\frac{\widehat{m}(T_R-1)}{\widehat{m}(0)}>\xi(T_R-1)\right]}\\
&\qquad + \mathbb{E}_{T_R-1}\left[\frac{\widehat{\boldsymbol{m}}(T_R)}{\widehat{m}^k(T_R-1)}\widehat{\boldsymbol{V}}_L(T_R)\right],
\end{aligned} \tag{IA57}$$

with

$$\widehat{c}_G^k(T_R-1) = \left(e^{\delta(T_R-2)} y \frac{\widehat{m}^k(T_R-1)}{\gamma_G \widehat{m}(1)}\right)^{-\frac{1}{1-\gamma_G}}, \tag{IA58}$$

$$\widehat{c}_L^k(T_R-1) = -\left(e^{\delta(T_R-2)} y \frac{\widehat{m}^k(T_R-1)}{\eta\gamma_L \widehat{m}(1)}\right)^{-\frac{1}{1-\gamma_L}}. \tag{IA59}$$

In the previous step, we have determined $\widehat{\boldsymbol{V}}_L(T_R)$ and $\widehat{\boldsymbol{V}}_G(T_R)$. Using OLS regressions, we now determine the coefficients $\eta_{0,G}^V(T_R)$, $\eta_{1,G}^V(T_R)$, $\eta_{2,G}^V(T_R)$, $\eta_{0,L}^V(T_R)$, $\eta_{1,L}^V(T_R)$ and $\eta_{2,L}^V(T_R)$ such that

$$\log \widehat{\boldsymbol{V}}_G(T_R) \approx \eta_{0,G}^V(T_R) + \eta_{1,G}^V(T_R)\log\left(-\widehat{\boldsymbol{Y}}_2(T_R)\right) + \eta_{2,G}^V(T_R)\log \widehat{\boldsymbol{m}}(T_R), \tag{IA60}$$

$$\begin{aligned}
\log\left(-\widehat{\boldsymbol{V}}_L(T_R)\right) &\approx \eta_{0,L}^V(T_R) + \eta_{1,L}^V(T_R)\log\left(-\widehat{\boldsymbol{Y}}_{2,L}(T_R)\right)\\
&\quad + \eta_{2,L}^V(T_R)\log \widehat{\boldsymbol{m}}(T_R).
\end{aligned} \tag{IA61}$$

Hence,

$$\widehat{V}^{i,j,k}(T_R-1) = \widehat{c}^{\,k}(T_R-1) + e^{\eta^V_{0,G}(T_R)} \left(-\widehat{Y}_2^j(T_R-1)\right)^{\eta^V_{1,G}(T_R)} \left(\widehat{m}^k(T_R-1)\right)^{\eta^V_{2,G}(T_R)}$$
$$\times\, \mathbb{E}_{T_R-1}\left[\left(\frac{\widehat{\boldsymbol{Y}}_2(T_R)}{\widehat{Y}_2^j(T_R-1)}\right)^{\eta^V_{1,G}(T_R)} \left(\frac{\widehat{\boldsymbol{m}}(T_R)}{\widehat{m}^k(T_R-1)}\right)^{\eta^V_{2,G}(T_R)+1}\right]$$
$$-\, e^{\eta^V_{0,L}(T_R)} \left(-\widehat{Y}_2^j(T_R-1)\right)^{\eta^V_{1,L}(T_R)} \left(\widehat{m}^k(T_R-1)\right)^{\eta^V_{2,L}(T_R)}$$
$$\times\, \mathbb{E}_{T_R-1}\left[\left(\frac{\widehat{\boldsymbol{Y}}_2(T_R)}{\widehat{Y}_2^j(T_R-1)}\right)^{\eta^V_{1,L}(T_R)} \left(\frac{\widehat{\boldsymbol{m}}(T_R)}{\widehat{m}^k(T_R-1)}\right)^{\eta^V_{2,L}(T_R)+1}\right].$$

We now find that

$$\begin{aligned}
\Delta\widehat{\boldsymbol{V}}(T_R) \approx\ & e^{\eta^V_{0,G}(T_R)} \left(-\widehat{Y}_2^j(T_R-1)\right)^{\eta^V_{1,G}(T_R)} \left(\widehat{m}^k(T_R-1)\right)^{\eta^V_{2,G}(T_R)} \\
& \times \Big\{ \ldots + \left(\eta^V_{1,G}(T_R)\sigma_{\bar{C}} + \eta^V_{2,G}(T_R)\widehat{\phi}^{i,j,k}_{\bar{C}}(T_R-1)\right)\epsilon_{\bar{C}}(T_R) \\
& \quad + \eta^V_{2,G}(T_R)\widehat{\phi}^{i,j,k}_Y(T_R-1)\epsilon_Y(T_R) + \eta^V_{2,G}(T_R)\widehat{\phi}^{i,j,k}_S(T_R-1)\epsilon_S(T_R)\Big\} \\
& - e^{\eta^V_{0,L}(T_R)} \left(-\widehat{Y}_2^j(T_R-1)\right)^{\eta^V_{1,L}(T_R)} \left(\widehat{m}^k(T_R-1)\right)^{\eta^V_{2,L}(T_R)} \\
& \times \Big\{ \ldots + \left(\eta^V_{1,L}(T_R)\sigma_{\bar{C}} + \eta^V_{2,L}(T_R)\widehat{\phi}^{i,j,k}_{\bar{C}}(T_R-1)\right)\epsilon_{\bar{C}}(T_R) \\
& \quad + \eta^V_{2,L}(T_R)\widehat{\phi}^{i,j,k}_Y(T_R-1)\epsilon_Y(T_R) + \eta^V_{2,L}(T_R)\widehat{\phi}^{i,j,k}_S(T_R-1)\epsilon_S(T_R)\Big\} \\
& - \widehat{c}^{\,k}(T_R-1).
\end{aligned} \tag{IA62}$$

We finally derive the change in optimal dual financial wealth. We find

$$\begin{aligned}
\Delta\widehat{\boldsymbol{F}}(T_R) \approx\ & \widehat{Y}_1^i(T_R-1) + \widehat{Y}_2^j(T_R-1) - \widehat{c}^{\,k}(T_R-1) \\
& + \widehat{F}^{i,j,k}(T_R-1)\left(\widehat{r}(T_R-1) + \widehat{\omega}^{i,j,k}(T_R-1)\lambda_S\sigma_S\right. \\
& \qquad \left. + \widehat{\omega}^{i,j,k}(T_R-1)\sigma_S\epsilon_S(T_R)\right).
\end{aligned} \tag{IA63}$$

Here,

$$
\begin{aligned}
\widehat{F}^{i,j,k}(T_R-1) = {} & e^{\eta_0^V(T_R)} \left(-\widehat{Y}_2^j(T_R-1)\right)^{\eta_1^V(T_R)} \left(\widehat{m}^k(T_R-1)\right)^{\eta_2^V(T_R)} \\
& + e^{\eta_0^{H_2}(T_R)} \left(-\widehat{Y}_2^j(T_R-1)\right)^{\eta_1^{H_2}(T_R)} \left(\widehat{m}^k(T_R-1)\right)^{\eta_2^{H_2}(T_R)} \\
& - \widehat{Y}_1^i(T_R-1).
\end{aligned} \tag{IA64}
$$

Using the condition $\Delta\widehat{\boldsymbol{V}}(T_R) = \Delta\widehat{\boldsymbol{H}}_1(T_R) + \Delta\widehat{\boldsymbol{H}}_2(T_R) + \Delta\widehat{\boldsymbol{F}}(T_R)$, we arrive at

$$
\begin{aligned}
\widehat{\omega}^{*i,j,k}(T_R-1) = {} & \eta_{2,G}^V(T_R)\widehat{\phi}_S^{i,j,k}(T_R-1)\frac{\widetilde{V}_G^{i,j,k}(T_R-1)}{\sigma_S\widehat{F}^{i,j,k}(T_R-1)} \\
& + \eta_{2,L}^V(T_R)\widehat{\phi}_S^{i,j,k}(T_R-1)\frac{\widetilde{V}_L^{i,j,k}(T_R-1)}{\sigma_S\widehat{F}^{i,j,k}(T_R-1)} \\
& - \eta_2^{H_2}(T_R)\widehat{\phi}_S^{i,j,k}(T_R-1)\frac{\widetilde{H}_2^{i,j,k}(T_R-1)}{\sigma_S\widehat{F}^{i,j,k}(T_R-1)},
\end{aligned} \tag{IA65}
$$

$$
\begin{aligned}
\widehat{\phi}_{\bar{C}}^{*i,j,k}(T_R-1) = {} & \sigma_{\bar{C}}\frac{\eta_{1,G}^V(T_R)\widetilde{V}_G^{i,j,k}(T_R-1) + \eta_{1,L}^V(T_R)\widetilde{V}_L^{i,j,k}(T_R-1)}{D^{i,j,k}(T_R-1,T_R)} \\
& - \sigma_{\bar{C}}\frac{\eta_1^{H_2}(T_R)\widetilde{H}_2^{i,j,k}(T_R-1)}{D^{i,j,k}(T_R-1,T_R)},
\end{aligned} \tag{IA66}
$$

$$
\widehat{\phi}_Y^{*i,j,k}(T_R-1) = -\frac{\sigma_Y\widehat{Y}_1^i(T_R-1)}{D^{i,j,k}(T_R-1,T_R)}, \tag{IA67}
$$

with

$$
\begin{aligned}
D^{i,j,k}(T_R-1,T_R) \equiv {} & \eta_2^{H_2}(T_R)\widetilde{H}_2^{i,j,k}(T_R-1) - \eta_{2,G}^V(T_R)\widetilde{V}_G^{i,j,k}(T_R-1) \\
& - \eta_{2,L}^V(T_R)\widetilde{V}_L^{i,j,k}(T_R-1),
\end{aligned} \tag{IA68}
$$

$$
\widetilde{H}_2^{i,j,k}(T_R-1) \equiv -e^{\eta_0^{H_2}(T_R)} \left(-\widehat{Y}_2^j(T_R-1)\right)^{\eta_1^{H_2}(T_R)} \left(\widehat{m}^k(T_R-1)\right)^{\eta_2^{H_2}(T_R)}, \tag{IA69}
$$

$$
\widetilde{V}_G^{i,j,k}(T_R-1) \equiv e^{\eta_{0,G}^V(T_R)} \left(-\widehat{Y}_2^j(T_R-1)\right)^{\eta_{1,G}^V(T_R)} \left(\widehat{m}^k(T_R-1)\right)^{\eta_{2,G}^V(T_R)}, \tag{IA70}
$$

$$
\widetilde{V}_L^{i,j,k}(T_R-1) \equiv -e^{\eta_{0,L}^V(T_R)} \left(-\widehat{Y}_2^j(T_R-1)\right)^{\eta_{1,L}^V(T_R)} \left(\widehat{m}^k(T_R-1)\right)^{\eta_{2,L}^V(T_R)}. \tag{IA71}
$$

Remaining Periods of the Working Phase

We now determine $\widehat{\omega}^{*i,j,k}(t)$, $\widehat{\lambda}_{\bar{C}}^{*i,j,k}(t)$ and $\widehat{\lambda}_{Y}^{*i,j,k}(t)$ for all $t \in \{1, \ldots, T_R - 2\}$. We start by deriving the change in the first and second part of dual human wealth. We find

$$\widehat{\boldsymbol{H}}_1(t+1) = \mathbb{E}_{t+1}\left[\sum_{l=0}^{T_R-t-1} \frac{\widehat{\boldsymbol{m}}(t+1+l)}{\widehat{\boldsymbol{m}}(t+1)}\widehat{\boldsymbol{Y}}_1(t+1+l)\right], \tag{IA72}$$

$$\begin{aligned}
\widehat{H}_1^{i,j,k}(t) &= \widehat{Y}_1^i(t) + \mathbb{E}_t\left[\sum_{l=0}^{T_R-t-1} \frac{\widehat{\boldsymbol{m}}(t+l+1)}{\widehat{m}^k(t)}\widehat{\boldsymbol{Y}}_1(t+l+1)\right] \\
&= \widehat{Y}_1^i(t) + \mathbb{E}_t\left\{\frac{\widehat{\boldsymbol{m}}(t+1)}{\widehat{m}^k(t)}\mathbb{E}_{t+1}\left[\sum_{l=0}^{T_R-t-1} \frac{\widehat{\boldsymbol{m}}(t+l+1)}{\widehat{\boldsymbol{m}}(t+1)}\widehat{\boldsymbol{Y}}_1(t+l+1)\right]\right\} \\
&= \widehat{Y}_1^i(t) + \mathbb{E}_t\left[\frac{\widehat{\boldsymbol{m}}(t+1)}{\widehat{m}^k(t)}\widehat{\boldsymbol{H}}_1(t+1)\right],
\end{aligned} \tag{IA73}$$

and

$$\widehat{\boldsymbol{H}}_2(t+1) = \mathbb{E}_{t+1}\left[\sum_{l=0}^{T_D-t-1} \frac{\widehat{\boldsymbol{m}}(t+1+l)}{\widehat{\boldsymbol{m}}(t+1)}\widehat{\boldsymbol{Y}}_2(t+1+l)\right], \tag{IA74}$$

$$\begin{aligned}
\widehat{H}_2^{i,j,k}(t) &= \widehat{Y}_2^j(t) + \mathbb{E}_t\left[\sum_{l=0}^{T_D-t-1} \frac{\widehat{\boldsymbol{m}}(t+l+1)}{\widehat{m}^k(t)}\widehat{\boldsymbol{Y}}_2(t+l+1)\right] \\
&= \widehat{Y}_2^j(t) + \mathbb{E}_t\left\{\frac{\widehat{\boldsymbol{m}}(t+1)}{\widehat{m}^k(t)}\mathbb{E}_{t+1}\left[\sum_{l=0}^{T_D-t-1} \frac{\widehat{\boldsymbol{m}}(t+l+1)}{\widehat{\boldsymbol{m}}(t+1)}\widehat{\boldsymbol{Y}}_2(t+l+1)\right]\right\} \\
&= \widehat{Y}_2^j(t) + \mathbb{E}_t\left[\frac{\widehat{\boldsymbol{m}}(t+1)}{\widehat{m}^k(t)}\widehat{\boldsymbol{H}}_2(t+1)\right].
\end{aligned} \tag{IA75}$$

In the previous step, we have determined $\widehat{\boldsymbol{H}}_1(t+1)$ and $\widehat{\boldsymbol{H}}_2(t+1)$. Using OLS regression, we now determine the coefficients $\eta_0^{H_1}(t+1)$, $\eta_0^{H_2}(t+1)$, $\eta_1^{H_1}(t+1)$, $\eta_1^{H_2}(t+1)$, $\eta_2^{H_1}(t+1)$,

$\eta_2^{H_2}(t+1)$, $\eta_3^{H_1}(t+1)$ and $\eta_3^{H_2}(t+1)$ such that

$$\begin{aligned}\log\left(\widehat{\boldsymbol{H}}_1(t+1)\right) \approx & \eta_0^{H_1}(t+1) + \eta_1^{H_1}(t+1)\log\left(-\widehat{\boldsymbol{Y}}_2(t+1)\right) \\ & + \eta_2^{H_1}(t+1)\log\widehat{\boldsymbol{m}}(t+1) + \eta_3^{H_1}(t+1)\log\widehat{\boldsymbol{Y}}_1(t+1),\end{aligned} \tag{IA76}$$

$$\begin{aligned}\log\left(-\widehat{\boldsymbol{H}}_2(t+1)\right) \approx & \eta_0^{H_2}(t+1) + \eta_1^{H_2}(t+1)\log\left(-\widehat{\boldsymbol{Y}}_2(t+1)\right) \\ & + \eta_2^{H_2}(t+1)\log\widehat{\boldsymbol{m}}(t+1) + \eta_3^{H_2}(t+1)\log\widehat{\boldsymbol{Y}}_1(t+1).\end{aligned} \tag{IA77}$$

Hence,

$$\begin{aligned}\widehat{H}_1^{i,j,k}(t) = & \widehat{Y}_1^i(t) + e^{\eta_0^{H_1}(t+1)}\left(-\widehat{Y}_2^j(t)\right)^{\eta_1^{H_1}(t+1)}\left(\widehat{m}^k(t)\right)^{\eta_2^{H_1}(t+1)}\left(\widehat{Y}_1^i(t)\right)^{\eta_3^{H_1}(t+1)} \\ & \times \mathbb{E}_t\left[\left(\frac{\widehat{\boldsymbol{Y}}_2(t+1)}{\widehat{Y}_2^j(t)}\right)^{\eta_1^{H_1}(t+1)}\left(\frac{\widehat{\boldsymbol{m}}(t+1)}{\widehat{m}^k(t)}\right)^{\eta_2^{H_1}(t+1)+1}\left(\frac{\widehat{\boldsymbol{Y}}_1(t+1)}{\widehat{Y}_1^i(t)}\right)^{\eta_3^{H_1}(t+1)}\right], \\ \widehat{H}_2^{i,j,k}(t) = & \widehat{Y}_2^j(t) - e^{\eta_0^{H_2}(t+1)}\left(-\widehat{Y}_2^j(t)\right)^{\eta_1^{H_2}(t+1)}\left(\widehat{m}^k(t)\right)^{\eta_2^{H_2}(t+1)}\left(\widehat{Y}_1^i(t)\right)^{\eta_3^{H_2}(t+1)} \\ & \times \mathbb{E}_t\left[\left(\frac{\widehat{\boldsymbol{Y}}_2(t+1)}{\widehat{Y}_2^j(t)}\right)^{\eta_1^{H_2}(t+1)}\left(\frac{\widehat{\boldsymbol{m}}(t+1)}{\widehat{m}^k(t)}\right)^{\eta_2^{H_2}(t+1)+1}\left(\frac{\widehat{\boldsymbol{Y}}_1(t+1)}{\widehat{Y}_1^i(t)}\right)^{\eta_3^{H_2}(t+1)}\right].\end{aligned}$$

We find that

$$
\begin{aligned}
\Delta\widehat{\boldsymbol{H}}_1(t+1) \approx\; & e^{\eta_0^{H_1}(t+1)} \left(-\widehat{Y}_2^j(t)\right)^{\eta_1^{H_1}(t+1)} \left(\widehat{m}^k(t)\right)^{\eta_2^{H_1}(t+1)} \left(\widehat{Y}_1^i(t)\right)^{\eta_3^{H_1}(t+1)} \\
& \times \Big\{ \ldots + \left(\eta_1^{H_1}(t+1)\sigma_{\bar{C}} + \eta_2^{H_1}(t+1)\widehat{\phi}_{\bar{C}}^{i,j,k}(t)\right)\epsilon_{\bar{C}}(t+1) \\
& \quad + \left(\eta_3^{H_1}(t+1)\sigma_Y + \eta_2^{H_1}(t+1)\widehat{\phi}_Y^{i,j,k}(t)\right)\epsilon_Y(t+1) \\
& \quad + \eta_2^{H_1}(t+1)\widehat{\phi}_S^{i,j,k}(t)\epsilon_S(t+1)\Big\} - \widehat{Y}_1^i(t),
\end{aligned}
\tag{IA78}
$$

$$
\begin{aligned}
\Delta\widehat{\boldsymbol{H}}_2(t+1) \approx\; & -e^{\eta_0^{H_2}(t+1)} \left(-\widehat{Y}_2^j(t)\right)^{\eta_1^{H_2}(t+1)} \left(\widehat{m}^k(t)\right)^{\eta_2^{H_2}(t+1)} \left(\widehat{Y}_1^i(t)\right)^{\eta_3^{H_2}(t+1)} \\
& \times \Big\{ \ldots + \left(\eta_1^{H_2}(t+1)\sigma_{\bar{C}} + \eta_2^{H_2}(t+1)\widehat{\phi}_{\bar{C}}^{i,j,k}(t)\right)\epsilon_{\bar{C}}(t+1) \\
& \quad + \left(\eta_3^{H_2}(t+1)\sigma_Y + \eta_2^{H_2}(t+1)\widehat{\phi}_Y^{i,j,k}(t)\right)\epsilon_Y(t+1) \\
& \quad + \eta_2^{H_2}(t+1)\widehat{\phi}_S^{i,j,k}(t)\epsilon_S(t+1)\Big\} - \widehat{Y}_2^j(t).
\end{aligned}
\tag{IA79}
$$

We now derive the change in the value of future dual consumption. We find

$$
\widehat{\boldsymbol{V}}_G(t+1) = \mathbb{E}_{t+1}\left[\sum_{l=0}^{T_D-t-1} \frac{\widehat{\boldsymbol{m}}(t+1+l)}{\widehat{\boldsymbol{m}}(t+1)}\widehat{\boldsymbol{c}}_G(t+1+l)\mathbb{1}_{\left[\frac{\widehat{\boldsymbol{m}}(t+1+l)}{\widehat{m}(0)}\le\xi(t+1+l)\right]}\right]
\tag{IA80}
$$

$$
\begin{aligned}
\widehat{V}_G^{j,k}(t) &= \widehat{c}_G^k(t)\mathbb{1}_{\left[\frac{\widehat{m}(t)}{\widehat{m}(0)}\le\xi(t)\right]} + \mathbb{E}_t\left[\sum_{l=0}^{T_D-t-1}\frac{\widehat{\boldsymbol{m}}(t+1+l)}{\widehat{m}^k(t)}\widehat{\boldsymbol{c}}_G(t+1+l)\right. \\
&\qquad \left.\mathbb{1}_{\left[\frac{\widehat{\boldsymbol{m}}(t+1+l)}{\widehat{m}(0)}\le\xi(t+1+l)\right]}\right] \\
&= \widehat{c}_G^k(t)\mathbb{1}_{\left[\frac{\widehat{m}(t)}{\widehat{m}(0)}\le\xi(t)\right]} + \mathbb{E}_t\left\{\frac{\widehat{\boldsymbol{m}}(t+1)}{\widehat{m}^k(t)}\mathbb{E}_{t+1}\left[\sum_{l=0}^{T_D-t-1}\frac{\widehat{\boldsymbol{m}}(t+1+l)}{\widehat{\boldsymbol{m}}(t+1)}\right.\right. \\
&\qquad \left.\left.\widehat{\boldsymbol{c}}_G(t+l+1)\mathbb{1}_{\left[\frac{\widehat{\boldsymbol{m}}(t+1+l)}{\widehat{m}(0)}\le\xi(t+1+l)\right]}\right]\right\} \\
&= \widehat{c}_G^k(t)\mathbb{1}_{\left[\frac{\widehat{m}(t)}{\widehat{m}(0)}\le\xi(t)\right]} + \mathbb{E}_t\left[\frac{\widehat{\boldsymbol{m}}(t+1)}{\widehat{m}^k(t)}\widehat{\boldsymbol{V}}_G(t+1)\right],
\end{aligned}
\tag{IA81}
$$

and

$$\widehat{\boldsymbol{V}}_L(t+1) = \mathbb{E}_{t+1}\left[\sum_{l=0}^{T_D-t-1}\frac{\widehat{\boldsymbol{m}}(t+1+l)}{\widehat{\boldsymbol{m}}(t+1)}\widehat{\boldsymbol{c}}_L(t+1+l)\mathbb{1}_{\left[\frac{\widehat{\boldsymbol{m}}(t+1+l)}{\widehat{m}(0)}>\xi(t+1+l)\right]}\right], \tag{IA82}$$

$$\begin{aligned}\widehat{V}_L^{j,k}(t) &= \widehat{c}_L^k(t)\mathbb{1}_{\left[\frac{\widehat{m}(t)}{\widehat{m}(0)}>\xi(t)\right]} + \mathbb{E}_t\left[\sum_{l=0}^{T_D-t-1}\frac{\widehat{\boldsymbol{m}}(t+1+l)}{\widehat{m}^k(t)}\widehat{\boldsymbol{c}}_L(t+1+l)\right.\\ &\qquad\left.\mathbb{1}_{\left[\frac{\widehat{\boldsymbol{m}}(t+1+l)}{\widehat{m}(0)}>\xi(t+1+l)\right]}\right]\\ &= \widehat{c}_L^k(t)\mathbb{1}_{\left[\frac{\widehat{m}(t)}{\widehat{m}(0)}>\xi(t)\right]} + \mathbb{E}_t\left\{\frac{\widehat{\boldsymbol{m}}(t+1)}{\widehat{m}^k(t)}\mathbb{E}_{t+1}\left[\sum_{l=0}^{T_D-t-1}\frac{\widehat{\boldsymbol{m}}(t+1+l)}{\widehat{\boldsymbol{m}}(t+1)}\right.\right.\\ &\qquad\left.\left.\widehat{\boldsymbol{c}}_L(t+l+1)\mathbb{1}_{\left[\frac{\widehat{\boldsymbol{m}}(t+1+l)}{\widehat{m}(0)}>\xi(t+1+l)\right]}\right]\right\}\\ &= \widehat{c}_L^k(t)\mathbb{1}_{\left[\frac{\widehat{m}(t)}{\widehat{m}(0)}>\xi(t)\right]} + \mathbb{E}_t\left[\frac{\widehat{\boldsymbol{m}}(t+1)}{\widehat{m}^k(t)}\widehat{\boldsymbol{V}}_L(t+1)\right],\end{aligned} \tag{IA83}$$

with

$$\widehat{c}_G^k(t) = \left(e^{\delta(t-1)}y\frac{\widehat{m}^k(t)}{\gamma_G\widehat{m}(1)}\right)^{-\frac{1}{1-\gamma_G}}, \tag{IA84}$$

$$\widehat{c}_L^k(t) = -\left(e^{\delta(t-1)}y\frac{\widehat{m}^k(t)}{\eta\gamma_L\widehat{m}(1)}\right)^{-\frac{1}{1-\gamma_L}}. \tag{IA85}$$

In the previous step, we have determined $\widehat{\boldsymbol{V}}_G(t+1)$ and $\widehat{\boldsymbol{V}}_L(t+1)$. Using OLS regressions, we now determine the coefficients $\eta_{0,G}^V(t+1)$, $\eta_{1,G}^V(t+1)$, $\eta_{2,G}^V(t+1)$, $\eta_{3,G}^V(t+1)$, $\eta_{0,L}^V(t+1)$, $\eta_{1,l}^V(t+1)$, $\eta_{2,l}^V(t+1)$ and $\eta_{3,L}^V(t+1)$ such that

$$\begin{aligned}\log\widehat{\boldsymbol{V}}_G(t+1) &\approx \eta_{0,G}^V(t+1) + \eta_{1,G}^V(t+1)\log\left(-\widehat{\boldsymbol{Y}}_2(t+1)\right)\\ &\quad + \eta_{2,G}^V(t+1)\log\widehat{\boldsymbol{m}}(t+1) + \eta_{3,G}^V(t+1)\log\widehat{\boldsymbol{Y}}_1(t+1),\\ \log\left(-\widehat{\boldsymbol{V}}_L(t+1)\right) &\approx \eta_{0,L}^V(t+1) + \eta_{1,L}^V(t+1)\log\left(-\widehat{\boldsymbol{Y}}_{2,L}(t+1)\right)\\ &\quad + \eta_{2,L}^V(t+1)\log\widehat{\boldsymbol{m}}(t+1) + \eta_{3,L}^V(t+1)\log\widehat{\boldsymbol{Y}}_1(t+1).\end{aligned} \tag{IA86}$$

Hence,

$$
\begin{aligned}
\widehat{V}^{i,j,k}(t) &= \widehat{c}^{\,k}(t) + e^{\eta^V_{0,G}(t+1)} \left(-\widehat{Y}^j_2(t)\right)^{\eta^V_{1,G}(t+1)} \left(\widehat{m}^k(t)\right)^{\eta^V_{2,G}(t+1)} \left(\widehat{Y}^i_1(t)\right)^{\eta^V_{3,G}(t+1)} \\
&\quad \times \mathbb{E}_t \left[\left(\frac{\widehat{\boldsymbol{Y}}_2(t+1)}{\widehat{Y}^j_2(t)}\right)^{\eta^V_{1,G}(t+1)} \left(\frac{\widehat{\boldsymbol{m}}(t+1)}{\widehat{m}^k(t)}\right)^{\eta^V_{2,G}(t+1)+1} \left(\frac{\widehat{\boldsymbol{Y}}_1(t+1)}{\widehat{Y}^i_1(t)}\right)^{\eta^V_{3,G}(t+1)} \right] \\
&\quad - e^{\eta^V_{0,L}(t+1)} \left(-\widehat{Y}^j_2(t)\right)^{\eta^V_{1,L}(t+1)} \left(\widehat{m}^k(t)\right)^{\eta^V_{2,L}(t+1)} \left(\widehat{Y}^i_1(t)\right)^{\eta^V_{3,L}(t+1)} \\
&\quad \times \mathbb{E}_t \left[\left(\frac{\widehat{\boldsymbol{Y}}_2(t+1)}{\widehat{Y}^j_2(t)}\right)^{\eta^V_{1,L}(t+1)} \left(\frac{\widehat{\boldsymbol{m}}(t+1)}{\widehat{m}^k(t)}\right)^{\eta^V_{2,L}(t+1)+1} \left(\frac{\widehat{\boldsymbol{Y}}_1(t+1)}{\widehat{Y}^i_1(t)}\right)^{\eta^V_{3,L}(t+1)} \right].
\end{aligned}
$$

We now find that

$$
\begin{aligned}
\Delta\widehat{\boldsymbol{V}}(t+1) &\approx e^{\eta^V_{0,G}(t+1)} \left(-\widehat{Y}^j_2(t)\right)^{\eta^V_{1,G}(t+1)} \left(\widehat{m}^k(t)\right)^{\eta^V_{2,G}(t+1)} \left(\widehat{Y}^i_1(t)\right)^{\eta^V_{3,G}(t+1)} \\
&\quad \times \Big\{ \ldots + \left(\eta^V_{1,G}(t+1)\sigma_{\bar{C}} + \eta^V_{2,G}(t+1)\widehat{\phi}^{i,j,k}_{\bar{C}}(t)\right) \epsilon_{\bar{C}}(t+1) \\
&\quad \times \Big\{ \ldots + \left(\eta^V_{3,G}(t+1)\sigma_{Y} + \eta^V_{2,G}(t+1)\widehat{\phi}^{i,j,k}_{Y}(t)\right) \epsilon_{Y}(t+1) \\
&\qquad + \eta^V_{2,G}(t+1)\widehat{\phi}^{i,j,k}_{S}(t)\epsilon_S(t+1) \Big\} \\
&\quad - e^{\eta^V_{0,L}(t+1)} \left(-\widehat{Y}^j_{2,L}(t)\right)^{\eta^V_{1,L}(t+1)} \left(\widehat{m}^k(t)\right)^{\eta^V_{2,L}(t+1)} \left(\widehat{Y}^i_1(t)\right)^{\eta^V_{3,L}(t+1)} \\
&\quad \times \Big\{ \ldots + \left(\eta^V_{1,L}(t+1)\sigma_{\bar{C}} + \eta^V_{2,L}(t+1)\widehat{\phi}^{i,j,k}_{\bar{C}}(t)\right) \epsilon_{\bar{C}}(t+1) \\
&\quad \times \Big\{ \ldots + \left(\eta^V_{3,L}(t+1)\sigma_{Y} + \eta^V_{2,L}(t+1)\widehat{\phi}^{i,j,k}_{Y}(t)\right) \epsilon_{Y}(t+1) \\
&\qquad + \eta^V_{2,L}(t+1)\widehat{\phi}^{i,j,k}_{S}(t)\epsilon_S(t+1) \Big\} - \widehat{c}^{\,k}(t).
\end{aligned} \tag{IA87}
$$

We finally derive the change in dual financial wealth. We find

$$
\begin{aligned}
\Delta\widehat{\boldsymbol{F}}(t+1) &\approx \widehat{Y}^i_1(t) + \widehat{Y}^j_2(t) - \widehat{c}^{\,k}(t) \\
&\quad + \widehat{F}^{i,j,k}(t)\left(\widehat{r}(t) + \widehat{\omega}^{i,j,k}(t)\lambda_S\sigma_S + \widehat{\omega}^{i,j,k}(t)\sigma_S\epsilon_S(t+1)\right).
\end{aligned} \tag{IA88}
$$

Here,

$$
\begin{aligned}
\widehat{F}^{i,j,k}(t) = {} & e^{\eta^V_{0,G}(t+1)} \left(-\widehat{Y}_2^j(t)\right)^{\eta^V_{1,G}(t+1)} \left(\widehat{m}^k(t)\right)^{\eta^V_{2,G}(t+1)} \left(\widehat{Y}_1^i(t)\right)^{\eta^V_{3,G}(t+1)} \\
& - e^{\eta^V_{0,L}(t+1)} \left(-\widehat{Y}_2^j(t)\right)^{\eta^V_{1,L}(t+1)} \left(\widehat{m}^k(t)\right)^{\eta^V_{2,L}(t+1)} \left(\widehat{Y}_1^i(t)\right)^{\eta^V_{3,L}(t+1)} \\
& + e^{\eta^{H_2}_0(t+1)} \left(-\widehat{Y}_2^j(t)\right)^{\eta^{H_2}_1(t+1)} \left(\widehat{m}^k(t)\right)^{\eta^{H_2}_2(t+1)} \left(\widehat{Y}_1^i(t)\right)^{\eta^{H_2}_3(t+1)} \\
& - e^{\eta^{H_1}_0(t+1)} \left(-\widehat{Y}_2^j(t)\right)^{\eta^{H_1}_1(t+1)} \left(\widehat{m}^k(t)\right)^{\eta^{H_1}_2(t+1)} \left(\widehat{Y}_1^i(t)\right)^{\eta^{H_1}_3(t+1)} .
\end{aligned}
\tag{IA89}
$$

Using the condition $\Delta\widehat{\boldsymbol{V}}(t+1) = \Delta\widehat{\boldsymbol{H}}_1(t+1) + \Delta\widehat{\boldsymbol{H}}_2(t+1) + \Delta\widehat{\boldsymbol{F}}(t+1)$, we arrive at

$$
\begin{aligned}
\widehat{\omega}^{*i,j,k}(t) = {} & \eta^V_{2,G}(t+1)\widehat{\phi}_S^{i,j,k}(t)\frac{\widetilde{V}_G^{i,j,k}(t)}{\sigma_S\widehat{F}^{i,j,k}(t)} + \eta^V_{2,L}(t+1)\widehat{\phi}_S^{i,j,k}(t)\frac{\widetilde{V}_L^{i,j,k}(t)}{\sigma_S\widehat{F}^{i,j,k}(t)} \\
& - \eta_2^{H_1}(t+1)\widehat{\phi}_S^{i,j,k}(t)\frac{\widetilde{H}_1^{i,j,k}(t)}{\sigma_S\widehat{F}^{i,j,k}(t)} - \eta_2^{H_2}(t+1)\widehat{\phi}_S^{i,j,k}(t)\frac{\widetilde{H}_2^{i,j,k}(t)}{\sigma_S\widehat{F}^{i,j,k}(t)},
\end{aligned}
\tag{IA90}
$$

$$
\begin{aligned}
\widehat{\phi}_{\bar{C}}^{*i,j,k}(t) = {} & \sigma_{\bar{C}}\frac{\eta^V_{1,G}(t+1)\widetilde{V}_G^{i,j,k}(t) + \eta^V_{1,L}(t+1)\widetilde{V}_L^{i,j,k}(t)}{D^{i,j,k}(t,t+1)} \\
& - \sigma_{\bar{C}}\frac{\eta_1^{H_1}(t+1)\widetilde{H}_1^{i,j,k}(t) + \eta_1^{H_2}(t+1)\widetilde{H}_2^{i,j,k}(t)}{D^{i,j,k}(t,t+1)},
\end{aligned}
\tag{IA91}
$$

$$
\begin{aligned}
\widehat{\phi}_{Y}^{*i,j,k}(t) = {} & \sigma_{Y}\frac{\eta^V_{3,G}(t+1)\widetilde{V}_G^{i,j,k}(t) + \eta^V_{3,L}(t+1)\widetilde{V}_L^{i,j,k}(t)}{D^{i,j,k}(t,t+1)} \\
& - \sigma_{Y}\frac{\eta_3^{H_1}(t+1)\widetilde{H}_1^{i,j,k}(t) + \eta_3^{H_2}(t+1)\widetilde{H}_2^{i,j,k}(t)}{D^{i,j,k}(t,t+1)},
\end{aligned}
\tag{IA92}
$$

with

$$D^{i,j,k}(t,t+1) \equiv \eta_2^{H_1}(t+1)\widetilde{H}_1^{i,j,k}(t) + \eta_2^{H_2}(t+1)\widetilde{H}_2^{i,j,k}(t) - \eta_{2,G}^{V}(t+1)\widetilde{V}_G^{i,j,k}(t) - \eta_{2,L}^{V}(t+1)\widetilde{V}_L^{i,j,k}(t), \tag{IA93}$$

$$\widetilde{H}_1^{i,j,k}(t) \equiv e^{\eta_0^{H_1}(t+1)}\left(-\widehat{Y}_2^j(t)\right)^{\eta_1^{H_1}(t+1)}\left(\widehat{m}^k(t)\right)^{\eta_2^{H_1}(t+1)}\left(\widehat{Y}_1^i(t)\right)^{\eta_3^{H_1}(t+1)}, \tag{IA94}$$

$$\widetilde{H}_2^{i,j,k}(t) \equiv -e^{\eta_0^{H_2}(t+1)}\left(-\widehat{Y}_2^j(t)\right)^{\eta_1^{H_2}(t+1)}\left(\widehat{m}^k(t)\right)^{\eta_2^{H_2}(t+1)}\left(\widehat{Y}_1^i(t)\right)^{\eta_3^{H_2}(t+1)}, \tag{IA95}$$

$$\widetilde{V}_G^{i,j,k}(t) \equiv e^{\eta_{0,G}^{V}(t+1)}\left(-\widehat{Y}_2^j(t)\right)^{\eta_{1,G}^{V}(t+1)}\left(\widehat{m}^k(t)\right)^{\eta_{2,G}^{V}(t+1)}\left(\widehat{Y}_1^i(t)\right)^{\eta_{3,G}^{V}(t+1)}, \tag{IA96}$$

$$\widetilde{V}_L^{i,j,k}(t) \equiv -e^{\eta_{0,L}^{V}(t+1)}\left(-\widehat{Y}_2^j(t)\right)^{\eta_{1,L}^{V}(t+1)}\left(\widehat{m}^k(t)\right)^{\eta_{2,L}^{V}(t+1)}\left(\widehat{Y}_1^i(t)\right)^{\eta_{3,L}^{V}(t+1)}. \tag{IA97}$$

We note that in applications $\eta_{3,G}^{V}(t+1)$, $\eta_{3,L}^{V}(t+1)$, $\eta_3^{H_2}(t+1)$, $\eta_2^{H_1}(t+1)$ and $\eta_2^{H_2}(t+1)$ are approximately equal to zero. Under these conditions, we thus find that

$$\widehat{\phi}_Y^{*i,j,k}(t) = \sigma_Y \frac{\eta_3^{H_1}(t+1)\widetilde{H}_1^{i,j,k}(t)}{\eta_{2,G}^{V}(t+1)\widetilde{V}_G^{i,j,k}(t) + \eta_{2,L}^{V}(t+1)\widetilde{V}_L^{i,j,k}(t)}, \tag{IA98}$$

which reduces to $-\widehat{\lambda}_Y^{*i,j,k}(t)$ if $\rho_{3,1} = \rho_{3,2} = 0$.