Generalized Orlicz premia*

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Abstract

We introduce a generalized version of Orlicz premia, based on possibly non-convex loss functions. We show that this generalized definition covers a variety of relevant examples, such as the geometric mean and the expectiles, while at the same time retaining a number of relevant properties. We establish that cash-additivity leads to L^p-quantiles, extending a classical result on 'collapse to the mean' for convex Orlicz premia.

We then focus on the geometrically convex case, discussing the dual representation of generalized Orlicz premia and comparing it with a multiplicative form of the standard dual representation for the convex case. Finally, we show that generalized Orlicz premia arise naturally as the only elicitable, positively homogeneous, monotone and normalized functionals.

Keywords Orlicz premia, expectiles, L^p-quantiles, geometric convexity, elicitability, return risk measures.

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1 Introduction

Orlicz premia were introduced in the actuarial literature by Haezendonck and Goovaerts in [28], the very first issue of *Insurance: Mathematics and Economics*. They constitute a class of extensively studied premium principles, defined by

$$H_\Phi(X) := \inf\{k > 0 | \,\mathbb{E}\,[\Phi(X/k)] \leqslant 1\},$$

where the loss (or Young) function $\Phi\colon [0, +\infty) \to [0, +\infty]$ is convex and satisfies $\Phi(0) = 0$ and $\Phi(1) = 1$. The standard actuarial interpretation is that H_Φ is a multiplicative version of the zero (or equivalent) utility premium principle that is positively homogeneous by construction and whose convexity is implied by the convexity of Φ. From a mathematical point of view, Orlicz premia are very attractive because the corresponding Orlicz (or Luxemburg) norms are a generalization of L^p-norms with a rich and well-understood duality theory. Furthermore, Orlicz premia are the basis for the definition of the Haezendonck-Goovaerts risk measures, introduced in [27] by a construction that resembles the notion of optimized certainty equivalent introduced by Ben-Tal and Teboulle in [11, 12] and that gives rise to a family of coherent risk measures whose properties have been widely studied. We refer to [6, 7, 13, 16, 17, 34, 9, 10] and the references therein for main results on Orlicz premia, applications of Orlicz spaces to risk measures and premium principles, and for Haezendonck-Goovaerts risk measures.

Our first step is to notice that a more general definition of Orlicz premia in which the requirement that the Young function Φ is convex and positive-valued is removed allows for the inclusion of many relevant examples, such as the geometric mean (or equivalently the logarithmic certainty equivalent) and the general family of generalized quantiles as in [8], of which the usual quantiles, expectiles and the L^p-quantiles introduced in [15] are prominent examples.

We show in Proposition 8 that the basic properties of Orlicz premia still hold in the generalized case, with the exception of convexity that essentially holds if and only if Φ is convex. Furthermore, we show in Theorem 9 that adding cash-additivity implies that the resulting Orlicz premia are necessarily L^p-quantiles, or expectiles if also convexity or concavity is assumed. This gives a novel point of view on expectiles as the only convex Orlicz premia that are also cash-additive, complementing Proposition 6 in [8] in which they are described as the only convex generalized quantiles that are positively homogeneous. Remarkably, Theorem 9 extends the classical result about 'collapse to the mean' of cash-additive, convex Orlicz premia (see [28, 26]) that is obtained under more restrictive assumptions on the Young function Φ to a more general 'collapse to expectiles'.

Removing the requirement of convexity of the Young function Φ allows for the consideration of the more general notion of geometric convexity, whose role in the axiomatic theory of risk measures parallels the one of the usual (arithmetic) convexity, in particular with respect to the class of the so-called return risk measures.[1]

[1] Recently, [9] introduced the class of return risk measures, consisting of normalized, mono-

In Proposition 14 we show that Orlicz premia are geometrically convex if and only if the Orlicz function Φ satisfies a suitable generalized convexity condition.

In Section 4 we present in Theorems 18, 21 and 24 a comparison of dual representations for convex and geometrically convex risk measures. Interestingly, in the geometrically convex case the dual representation takes the form of a multiplicatively penalized supremum of geometric means, to be compared with the multiplicatively penalized supremum of arithmetic means in the convex case. This result can be seen as a special case of a general duality theory for geometrically convex functions introduced in [3] that goes beyond the positively homogeneous case of return risk measures. The special case of Orlicz premia is then studied in Proposition 19 and Corollary 22.

Finally, in Section 5 we provide an alternative to the axiomatization of Orlicz premia given in Theorem 2 of [9] as the only law-invariant return risk measures whose level sets are convex with respect to mixtures, i.e., satisfying the so-called CxLS property. This is relevant because the CxLS property is a necessary condition for elicitability, as has been extensively discussed in the literature (see, e.g., [39, 25, 5, 19, 45, 21]). The main results given here in Theorem 28 and Corollary 29 under the assumption of either geometric convexity or convexity allow for the removal of a few technical hypotheses in [9].

The paper is organized as follows. In Section 2 we introduce generalized Orlicz premia and discuss their properties. In Section 3 we focus on the geometrically convex case. In Section 4 we discuss and compare dual representations in the convex and geometrically convex cases. In Section 5 we provide the axiomatization of convex and geometrically convex Orlicz premia based on the CxLS property. Section 6 concludes. All proofs are in the Appendix.

2 A generalization of Orlicz premia

The natural domain of Orlicz premia, for random variables $X : \Omega \to \mathbb{R}$ representing losses on a probability space $(\Omega, \mathcal{F}, P)$, is the Orlicz space

$$L^{\Phi} := \{X \in L^0(\Omega, \mathcal{F}) \mid \mathbb{E}[\Phi(X/k) \leqslant 1] \text{ for some } k > 0\}. \tag{1}$$

An important role in this definition is played by the convexity of the Young function Φ that implies that H_{Φ} is indeed a norm and L^{Φ} a Banach space.

Our starting point is to notice that the definition of Orlicz premia can be extended by allowing also for non-convex and non-positive-valued Young functions. From a mathematical point of view, the possibility of such an extension is mentioned in Chapter 10 of [40], leading to the notion of generalized Orlicz

tone and positively homogeneous functionals. Return risk measures provide relative (or geometric) assessments of risk, evaluating how much additional riskless log-return makes a financial position acceptable—whence their name. They constitute the relative counterparts of the class of monetary risk measures ([22, 18]), reminiscent of how relative risk aversion relates to absolute risk aversion. Their dynamic extensions have been studied in [10]; return risk measures that allow for probability distortion were recently analyzed in [44]; star-shaped generalizations were recently studied in [29, 30, 31, 32]; applications of return risk measures to capital allocation can be found in [35] and [14]; see also [4].

spaces, of which the L^p-spaces with $0 < p < 1$ are the simplest case. Interestingly, the generalized definition retains a number of remarkable properties and allows the inclusion within the enlarged family of generalized Orlicz premia of the following important examples.

Example 1 (Geometric mean) *Let* $\Phi(x) = 1 + \log(x)$. *Then,*

$$H_\Phi(X) = \inf\{k > 0 \mid \mathbb{E}[1 + \log(X/k)] \leqslant 1\} = \exp(\mathbb{E}[\log X]),$$

which is the geometric mean of X. *Here* Φ *is concave and* $\Phi(0) = -\infty$.

Example 2 (L^p-norm with $0 < p < 1$) *Let* $\Phi(x) = x^p$, *with* $0 < p < 1$. *Then,* $H_\Phi(X) = \|X\|_p$. *Here* Φ *is concave with* $\Phi(0) = 0$ *and* $\Phi(1) = 1$.

Example 3 (Quantiles) *Let* $0 < \alpha \leqslant 1$ *and*

$$\Phi_\alpha(x) = \begin{cases} \alpha & \text{if } 0 \leqslant x \leqslant 1, \\ 1 + \alpha & \text{if } x > 1. \end{cases}$$

Then,

$$\begin{aligned} H_{\Phi_\alpha}(X) &= \inf\{k > 0 \mid \mathbb{E}[\Phi_\alpha(X/k)] \leqslant 1\} \\ &= \inf\{k > 0 \mid \alpha P(X \leqslant k) + (1 + \alpha)(1 - P(X \leqslant k) \leqslant 1\} \\ &= \inf\{k > 0 \mid P(X \leqslant k) \geqslant \alpha\}, \end{aligned}$$

which is the left α*-quantile of* X. *Notice that if* $\alpha = 1$ *then* $H_{\Phi_\alpha}(X) = \operatorname{ess\,sup}(X)$. *Here* Φ_α *is neither convex nor concave and satisfies* $\Phi_\alpha(0) > 0$ *and* $\Phi_\alpha(1) < 1$.

Example 4 (Expectiles) *Let* $0 < \alpha < 1$ *and* $\Phi_\alpha(x) = 1 + \alpha(x - 1)_+ - (1 - \alpha)(x - 1)_-$. *Then,*

$$\begin{aligned} H_{\Phi_\alpha}(X) &= \inf\{k > 0 \mid \mathbb{E}[\Phi_\alpha(X/k)] \leqslant 1\} \\ &= \inf\{k > 0 \mid 1 + \alpha\,\mathbb{E}[(X/k - 1)_+ - (1 - \alpha)(X/k - 1)_-] \leqslant 1\} \\ &= \inf\{k > 0 \mid \alpha\,\mathbb{E}[(X - k)_+] \leqslant (1 - \alpha)\,\mathbb{E}[(X - k)_-]\} \\ &= e_\alpha(X), \end{aligned}$$

which is the α*-expectile of* X *introduced in [36]. Here* $\Phi_\alpha(0) > 0$ *and* Φ_α *is convex if* $1/2 \leqslant \alpha < 1$ *and concave if* $0 < \alpha \leqslant 1/2$.

Example 5 (L^p-quantiles) *Let* $0 < \alpha < 1$, $p > 0$ *and* $\Phi_{\alpha,p}(x) = 1 + \alpha(x - 1)_+^p - (1 - \alpha)(x - 1)_-^p$. *Then, as before,*

$$\begin{aligned} H_{\Phi_\alpha}(X) &= \inf\{k > 0 \mid \mathbb{E}[\Phi_{\alpha,p}(X/k)] \leqslant 1\} \\ &= \inf\left\{k > 0 \mid \alpha\,\mathbb{E}\left[(X - k)_+^p\right] \leqslant (1 - \alpha)\,\mathbb{E}[(X - k)_-^p]\right\}, \end{aligned}$$

which is an L^{p+1}*-quantile as defined in [15].*

These examples illustrate that removing the convexity and the normalization assumptions on Φ produces several relevant examples that have a number of relevant properties. We refer to these possibly non-convex Φ as Orlicz functions in the definition below in order to distinguish them from classical Young functions.

Definition 6 *An Orlicz function* $\Phi\colon [0,+\infty) \to \mathbb{R} \cup \{\pm\infty\}$ *satisfies:*

a) $\Phi(x) > -\infty$ *if* $x > 0$, $\Phi(x) \leqslant 1$ *if* $x \leqslant 1$, $\Phi(x) > 1$ *if* $x > 1$

b) Φ *is nondecreasing*

c) Φ *is left-continuous.*

If Φ is not convex, the set L^Φ introduced in (1) is not necessarily a vector space anymore, so we will consider $L^\infty_+ := \{X \in L^\infty \mid X \geqslant 0\ P\text{-a.s.}\}$ as domain.

Definition 7 *For* $X \in L^\infty_+$, *the Orlicz premium is defined by*

$$H_\Phi(X) = \inf\{k > 0 \mid \mathbb{E}\left[\Phi(X/k)\right] \leqslant 1\}.$$

If $P(X = 0) > 0$ *and* $\Phi(0) = -\infty$, *we set by definition* $H_\Phi(X) = 0$.

The properties required to the Orlicz function Φ in Definition 6 are necessary to preserve the fundamental properties of Orlicz premia, with the exception of convexity. Henceforth, equalities and inequalities between random variables are meant to hold P-a.s. without further mentioning.

Proposition 8 *Let* Φ *and* $H_\Phi(X)$ *be as in Definitions 6 and 7. Then:*

a) H_Φ *is monotone (i.e.,* $X \leqslant Y \Rightarrow H_\Phi(X) \leqslant H_\Phi(Y)$*), positively homogeneous (i.e.,* $H_\Phi(\lambda X) = \lambda H_\Phi(X)$, $\forall \lambda \geqslant 0$*) and satisfies* $H_\Phi(1) = 1$

b) If $u := \sup\{x \mid \Phi(x) < +\infty\}$, *then* $\frac{\operatorname{ess\,sup} X}{u} \leqslant H_\Phi(X) \leqslant \operatorname{ess\,sup} X$

c) If $H_\Phi(X) > 0$, *then* $H_\Phi(X) = \min\{k > 0 | \mathbb{E}[\Phi(X/k)] \leqslant 1\}$

d) It holds that $H_\Phi(X) \leqslant 1 \iff \mathbb{E}[\Phi(X)] \leqslant 1$

e) If Φ *is finite, strictly increasing and continuous, then* H_Φ *is the unique solution of the equation* $\mathbb{E}\left[\Phi\left(X/H_\Phi\right)\right] = 1$

f) If Φ *is convex, then* H_Φ *is convex (i.e.,* $H_\Phi(\alpha X + (1-\alpha)Y) \leqslant \alpha H_\Phi(X) + (1-\alpha)H_\Phi(Y)$, $\forall \alpha \in (0,1)$*))*

g) If there exists $u_1 < 1$ *with* $\Phi(u_1) < 1$ *and* $u_2 > 1$ *with* $\Phi(u_2) < +\infty$, *then* H_Φ *is convex only if* Φ *is convex*

h) H_Φ *is law-invariant in the sense that* $X \overset{d}{=} Y \Rightarrow H_\Phi(X) = H_\Phi(Y)$.

If the Orlicz function Φ does not satisfy left-continuity, then properties c) and d) in Proposition 8 may not hold. Notice also that $H_\Phi(X) = 1$ does not imply $\mathbb{E}[\Phi(X)] = 1$. A classical result about convex Orlicz premia is that they are cash-additive (i.e., $H_\Phi(X + h) = H_\Phi(X) + h$, $\forall h \in \mathbb{R}$) if and only if they collapse to the mean, as has been proved in [28, 26] under the additional assumption that the Young function Φ is differentiable. Remarkably, enlarging the class of loss functions as in Definition 6 enlarges the class of cash-additive Orlicz premia as established in the following.

Theorem 9 *Let Φ and $H_\Phi(X)$ be as in Definitions 6 and 7 with Φ finite, continuous and strictly increasing. Then:*

a) If H_Φ is cash-additive, then there exist $a > 0$, $b > 0$ and $p \geqslant 0$ such that

$$\Phi(x) = 1 + a(x-1)_+^p - b(x-1)_-^p.$$

b) If H_Φ is cash-additive and convex (resp. concave), then

$$\Phi(x) = 1 + a(x-1)_+ - b(x-1)_-,$$

with $a \geqslant b$ (resp. $b \leqslant a$).

In both cases, letting $\alpha = a/(a+b)$ leads to Examples 3, 4 and 5, so we can conclude that a cash-additive generalized Orlicz premium is an L^p-quantile that is necessarily an expectile if additionally convexity or concavity holds.

Example 10 (**$L^{p,q}$-quantiles**) *An interesting generalization of Example 5 is given by the family of Orlicz functions*

$$\Phi(x) = 1 + a(x-1)_+^p - b(x-1)_-^q,$$

with $a, b \geqslant 0$ and $p, q \geqslant 1$. As a consequence of Theorem 9, cash additivity holds if and only if $p = q$, as can be verified directly. Here, Φ is convex if and only if $a \geqslant b$ and $p = q = 1$. Note that the $L^{p,q}$-quantile is cash-subadditive if $p \geqslant q$ and cash-superadditive if $p \leqslant q$. Suppose that $H_\Phi(X) = k$, then e) in Proposition 8 implies that

$$a\,\mathbb{E}\left[\left(\frac{X-k}{k}\right)_+^p\right] = b\,\mathbb{E}\left[\left(\frac{X-k}{k}\right)_-^q\right].$$

For any $m \geqslant 0$, assuming $p \geqslant q$ we have $\left(\frac{m+k}{k}\right)^{p-q} \geqslant 1$, which gives

$$\begin{aligned}
&a\,\mathbb{E}\left[\left(\frac{X-k}{k}\right)_+^p\right] \leqslant b\,\mathbb{E}\left[\left(\frac{X-k}{k}\right)_-^q\right]\left(\frac{m+k}{k}\right)^{p-q} \\
&\Rightarrow a\,\mathbb{E}\left[\left(\frac{X-k}{m+k}\right)_+^p\right] \leqslant b\,\mathbb{E}\left[\left(\frac{X-k}{m+k}\right)_-^q\right] \\
&\Rightarrow a\,\mathbb{E}\left[\left(\frac{X+m-\tilde{k}}{\tilde{k}}\right)_+^p\right] \leqslant b\,\mathbb{E}\left[\left(\frac{X+m-\tilde{k}}{\tilde{k}}\right)_-^q\right],
\end{aligned}$$

where $\tilde{k} = k + m$. By the definition of Orlicz premia, we have

$$H_\Phi(X+m) \leqslant \tilde{k} = H_\Phi(X) + m.$$

The case $p \leqslant q$ follows by a similar argument.

3 Geometric convexity

Having relaxed the assumption of convexity on Φ and H_Φ allows us to study other, more general forms of convexity. We start by recalling the definitions of geometric convexity (called GG-convexity in the sequel) and geometric-arithmetic convexity (called GA-convexity) for functions of a single variable.

Definition 11 *A function $f\colon [0,+\infty) \to [0,+\infty]$ is GG-convex if for each $x, y \in [0,+\infty)$ and $\lambda \in (0,1)$ it holds that*

$$f(x^\lambda y^{1-\lambda}) \leqslant f^\lambda(x) f^{1-\lambda}(y),$$

where we set $0 \cdot +\infty = +\infty$ by definition.
A function $f\colon [0,+\infty) \to [-\infty,+\infty]$ is called GA-convex if for each $x, y \in [0,+\infty)$ and $\lambda \in (0,1)$ it holds that

$$f(x^\lambda y^{1-\lambda}) \leqslant \lambda f(x) + (1-\lambda) f(y),$$

where we set $-\infty + \infty = +\infty$ by definition.

GG and GA-convexity are types of algebraic convexity owing their name to the presence of both the Geometric and the Arithmetic means in their definitions. We refer to e.g., [37, 38] for further properties of these functions. As for the case of the usual convexity, the definitions of GG-convexity and GA-convexity for functionals such as premium principles and risk measures are similar. More specifically, in the following we consider premium principles or risk measures defined on $L^\infty(\Omega, \mathcal{F}, P)$ or on its subsets L^∞_+ and $L^\infty_{++} := \{X \in L^\infty_+ \mid X \geqslant c > 0\ P\text{-a.s.}\}$, where $(\Omega, \mathcal{F}, P)$ is a fixed nonatomic probability space.

Definition 12 *A functional $\rho\colon L^\infty_+ \to [0,\infty]$ is GG-convex if for each $X, Y \in L^\infty_+$ and $\lambda \in (0,1)$*

$$\rho(X^\lambda Y^{1-\lambda}) \leqslant \rho^\lambda(X) \rho^{1-\lambda}(Y),$$

where we set $0 \cdot (+\infty) = +\infty$ by definition. A functional $\rho\colon L^\infty_+ \to [-\infty,+\infty]$ is GA-convex if for each $X, Y \in L^\infty_+$ and $\lambda \in (0,1)$

$$\rho(X^\lambda Y^{1-\lambda}) \leqslant \lambda \rho(X) + (1-\lambda)\rho(Y),$$

where we set $\infty - \infty = +\infty$ by definition.

In the next proposition, we recall some elementary implications among the various forms of convexity that essentially follow from the AM-GM inequality. We refer to [3] for further discussions on this topic.

Proposition 13 *Let f and ρ be as in Definitions 11 and 12. Then:*

a) f is GG-convex if and only if $\log f(e^x)$ is convex

b) f is GA-convex if and only if $f(e^x)$ is convex *c) If f is GG-convex then it is GA-convex*

d) If f is nondecreasing and convex then it is GA-convex

e) If ρ is monotone, positively homogeneous and convex then it is GG-convex.

From Proposition 8 items f) and g) we know that the convexity of Φ is equivalent to the convexity of H_Φ, under very general assumptions on Φ. The next proposition shows that a similar characterization applies to GG-convexity.

Proposition 14 *Let Φ and $H_\Phi(X)$ be as in Definitions 6 and 7. If Φ is GA-convex, then H_Φ is GG-convex. Conversely, if there exist $u_1 < 1$ with $\Phi(u_1) < 1$ and $u_2 > 1$ with $\Phi(u_2) < +\infty$, then H_Φ is GG-convex only if Φ is GA-convex.*

So, under general assumptions on Φ, the GG-convexity of H_Φ corresponds to the GA-convexity of Φ. This is consistent with items e) and d) in Proposition 13, which imply that the class of geometrically convex Orlicz premia is strictly larger than the class of Orlicz premia, and that the corresponding class of nondecreasing and GA-convex Orlicz functions is strictly larger than the class of nondecreasing and convex Orlicz functions, respectively. Reconsidering the examples at the beginning of Section 2, it is not difficult to verify that the Orlicz function Φ is GA-convex in Examples 1, 2, and in Example 10 in the case $a \geqslant b$ and $p = q = 1$ that corresponds to convex expectiles, while it does not satisfy the GA-convexity property in all the other cases. As a corollary of Proposition 14, we obtain the following stronger formulation of item b) in Theorem 9.

Corollary 15 *Let Φ and $H_\Phi(X)$ be as in Definitions 6 and 7 with Φ finite, continuous and strictly increasing. If H_Φ is cash-additive and geometrically convex (resp. concave), then $\Phi(x) = 1 + a(x-1)_+ - b(x-1)_-$, with $a \geqslant b$ (resp. $b \leqslant a$).*

Convex expectiles are thus the only geometrically convex and cash-additive Orlicz premia. That is, in the presence of cash-additivity, the distinction between geometric convexity and the usual convexity for Orlicz premia is non-existent, but when relaxing cash-additivity, the class of geometrically convex Orlicz premia is strictly larger than that of convex Orlicz premia. Another interesting example representative of a large class of GG-convex functionals is the following.

Example 16 (Geometric expectiles) *Let $\Phi(x) = 1 + a(\log x)_+ - b(\log x)_-$, with $a, b \geqslant 0$. Then,*

$$\begin{aligned} H_\Phi(X) &= \inf\{k > 0 \mid \mathbb{E}[a(\log X - \log k)_+ - b(\log X - \log k)_-] \leqslant 0\} \\ &= \exp((\inf\{u \in \mathbb{R} \mid \mathbb{E}[a(\log X - u)_+ - b(\log X - u)_-] \leqslant 0\}) \\ &= \exp\left(e_\alpha(\log X)\right), \end{aligned}$$

where $\alpha = a/(a+b)$, which we call the geometric α-expectile. From Proposition 13 items b) and a) it is immediate to verify that if $a \geqslant b$ then Φ is GA-convex and H_Φ is GG-convex, as in the thesis of Proposition 14.

4 Dual representations

In this section, we study duality formulas for convex and geometrically convex Orlicz premia, and we discuss some issues in the construction of Haezendonck-Goovaerts risk measures in the non-convex case. We first recall from [9] the notion of a return risk measure, and prove two general results about the dual representation of convex and geometrically convex return risk measures.

Definition 17 *A* return risk measure $\rho\colon L^\infty_+ \to [0,+\infty)$ *is a positively homogeneous and monotone risk measure satisfying* $\rho(1) = 1$*. Its multiplicative acceptance set is* $B_\rho = \{X \in L^\infty_+ \mid \rho(X) \leqslant 1\}$*. Furthermore,* ρ *satisfies the Fatou property if* $X_n \xrightarrow{P} X$, $\|X_n\|_\infty \leqslant k \implies \rho(X) \leqslant \liminf_{n\to+\infty} \rho(X_n)$*, and the Lebesgue property if* $X_n \xrightarrow{P} X$, $\|X_n\|_\infty \leqslant k \implies \rho(X_n) \to \rho(X)$.

If P, Q are probability measures on $(\Omega, \mathcal{F})$, we say that Q is absolutely continuous with respect to P and we write $Q \ll P$ if, $\forall A \in \mathcal{F}$, $P(A) = 0 \Rightarrow Q(A) = 0$. In the following, we denote by $\mathbf{P}$ the set of probability measures on $(\Omega, \mathcal{F})$ that are absolutely continuous with respect to the reference measure P, with $\varphi_Q = \mathrm{d}Q/\mathrm{d}P$ the Radon-Nikodym derivative of Q with respect to P and with $q_X(\alpha)$ any version of the quantile function of the real random variable X.

Theorem 18 *Let* $\rho\colon L^\infty_+ \to [0,+\infty)$ *be a convex return risk measure satisfying the Fatou property. Then,*

$$\rho(X) = \sup_{Q\in\mathbf{P}} \{\beta(Q)\, \mathbb{E}_Q[X]\}, \tag{2}$$

where $\beta\colon \mathbf{P} \to [0,1]$ *is given by*

$$\beta(Q) = \Big[\sup_{X\in B_\rho} \mathbb{E}_Q[X]\Big]^{-1}, \tag{3}$$

where B_ρ *is the multiplicative acceptance set of* ρ *introduced in Definition 17. If* ρ *satisfies the Lebesgue property, then the supremum in* (2) *is attained. Furthermore, if* ρ *is law-invariant then*

$$\rho(X) = \sup_{Q\in\mathbf{P}} \left\{\beta(Q) \int_0^1 q_X(t) q_{\varphi_Q}(t)\,\mathrm{d}t\right\},$$

where

$$\beta(Q) = \left[\sup_{X\in B_\rho} \int_0^1 q_X(t) q_{\varphi_Q}(t)\mathrm{d}t\right]^{-1}.$$

The interpretation is straightforward: any convex return risk measure satisfying the Fatou property can be represented as a multiplicatively weighted supremum of expectations with respect to different probabilistic models $Q \in \mathbf{P}$. The 'discount factor' $\beta(\cdot)$ can be interpreted as an index of model plausibility under ambiguity. This representation is a multiplicative version of the classical one for cash-additive convex risk measures in [24, 23]. In the specific case of convex Orlicz premia, the duality formula admits a more explicit representation involving the convex conjugate of the Orlicz function Φ, defined by $\Psi(y) := \sup_{x\geqslant 0}\{xy - \Phi(x)\}$.

Proposition 19 *Let Φ and H_Φ be as in Definition 6, with Φ convex. Then,* (2) *holds with*

$$\beta(Q) = \left(\inf_{\lambda>0} \frac{1}{\lambda}\,\mathbb{E}\left[1 + \Psi\left(\lambda\frac{\mathrm{d}Q}{\mathrm{d}P}\right)\right]\right)^{-1}, \tag{4}$$

where Ψ is the convex conjugate of Φ defined above.

This result appears to have a nice interpretation in terms of Orlicz space theory. From Theorem 13 on p. 69 of [40], recalling that $\varphi_Q = dQ/dP$, it follows that

$$\beta(Q) = \left(\|\varphi_Q\|_\Psi\right)^{-1},$$

where $\|X\|_\Psi := \sup\{\mathbb{E}[XY] \mid H_\Phi(Y) \leqslant 1\}$ is the so-called Orlicz norm.

In Proposition 13, item e), we showed that a convex return risk measure is also geometrically convex, so we move now to the derivation of an alternative and more general dual representation for the geometrically convex case. To this aim, the Fatou and Lebesgue properties have to be slightly modified as follows.

Definition 20 *We say that ρ satisfies the* lower-bounded Fatou property *if*

$$X_n \xrightarrow{P} X,\ \|X_n\|_\infty \leqslant k, X_n \geqslant c > 0 \implies \rho(X) \leqslant \liminf_{n\to+\infty} \rho(X_n),$$

and the lower-bounded Lebesgue property *if*

$$X_n \xrightarrow{P} X,\ \|X_n\|_\infty \leqslant k, X_n \geqslant c > 0 \implies \rho(X_n) \to \rho(X).$$

Theorem 21 *Let $\rho\colon L^\infty_{++} \to (0,+\infty)$ be a geometrically convex return risk measure satisfying the lower-bounded Fatou property. Then,*

$$\rho(X) = \sup_{Q\in\mathbf{P}}\{\alpha(Q)\exp\left(\mathbb{E}_Q[\log X]\right)\}, \tag{5}$$

where $\alpha\colon \mathbf{P} \to [0,1]$ is given by

$$\alpha(Q) = \left[\sup_{X\in B_\rho} \exp(\mathbb{E}_Q[\log X])\right]^{-1}, \tag{6}$$

with B_ρ the multiplicative acceptance set of ρ. If ρ satisfies the lower-bounded Lebesgue property, then the supremum in (5) *is attained. Furthermore, if ρ is law-invariant, then*

$$\rho(X) = \sup_{Q\in\mathbf{P}}\left\{\exp\left(\alpha(Q)\int_0^1 q_{\log X}(t)q_{\varphi_Q}(t)\,\mathrm{d}t\right)\right\},$$

where

$$\alpha(Q) = \left[\sup_{X\in B_\rho} \exp\left(\int_0^1 q_{\log X}(t)q_{\varphi_Q}(t)\,\mathrm{d}t\right)\right]^{-1}.$$

In comparison with (2), the basic building block of (5) is the geometric mean. This theorem shows that any geometrically convex return risk measure can be expressed as the supremum of multiplicatively weighted geometric means, computed with respect to different probability measures $Q \in \mathbf{P}$. The analogy with (2) in Proposition 18 is evident: here the geometric mean replaces the usual (artihmetic) expectation. Incidentally, we mention that in the related paper [3] we show that dual representations similar to (5) and (6) hold for general geometrically convex functions, also without the monotonicity and the positive homogeneity assumptions, and are instances of a generally defined geometrically convex transform whose properties parallel those of the Fenchel transform. In the specific case of geometrically convex Orlicz premia, from Proposition 8, item d), we immediately get the following.

Corollary 22 *Let Φ and H_Φ be as in Definitions 6 and 7, with Φ GA-convex. Then, equation* (5) *holds with*

$$\alpha(Q) = \left[\sup_{\mathbb{E}[\Phi(X)] \leqslant 1} \exp(\mathbb{E}[\log X]) \right]^{-1}.$$

From Proposition 13, item e), it follows that the dual representation given in Theorem 21 is a generalization of the one given in Proposition 18. Interestingly, it is possible to establish an explicit link between the multiplicative penalties $\alpha(Q)$ in equation (5) of Theorem 21 and $\beta(Q)$ in equation (2) of Theorem 18. Recall the definition of relative entropy or Kullback-Leibler divergence.

Definition 23 *If R, Q are probability measures on (Ω, P), then the relative entropy of R with respect to Q is given by*

$$\mathrm{H}(R, Q) := \begin{cases} \mathbb{E}_Q \left[\frac{\mathrm{d}R}{\mathrm{d}Q} \log \frac{\mathrm{d}R}{\mathrm{d}Q} \right] & \text{if } R \ll Q \\ +\infty & \text{otherwise.} \end{cases}$$

Proposition 24 *Let $\rho \colon L^\infty_{++} \to (0, +\infty)$ be a convex return risk measure with a dual representation given as in* (2)*. Then, ρ admits a dual representation as in* (5) *with*

$$\alpha(R) = \sup_{Q \ll P} \left\{ \frac{\beta(Q)}{\exp(\mathrm{H}(R, Q))} \right\}.$$

We end the section by recalling the definition of the Haezendonck-Goovaerts (HG) risk measure given in [27] (see also [6, 9, 10] and the references therein), i.e.,

$$\rho_{\mathrm{HG}}(X) = \inf_{x \in \mathbb{R}} \left\{ x + H_\Phi \left((X - x)_+ \right) \right\}. \tag{7}$$

The idea of this construction goes back to the notion of optimized certainty equivalent introduced in [11] and [12] and is, in fact, applicable to *any* return risk measure, also beyond the case of generalized Orlicz premia, thus yielding 'optimized return (OR) risk measures'. From the definition, it follows immediately that ρ_{HG} is cash-additive, and inherits from H_Φ the properties of monotonicity, positive homogeneity and convexity, so, when Φ is convex, ρ_{HG} is a coherent risk measure. Its dual set has been derived in Proposition 4 of [7] in terms of the Orlicz norm, and its dual representation can be simply written with the notations of Proposition 18 by

$$\rho_{\mathrm{HG}}(X) = \sup_{Q \in \mathbf{Q}} \mathbb{E}_Q[X],$$

where $\mathbf{Q} := \{Q \in \mathbf{P} \mid \tilde{\beta}(Q) = 1\}$. Indeed, as has been observed in [7], the HG risk measure can be seen as the inf-convolution of H_Φ with the functional

$$g(X) := \begin{cases} x & \text{if } X = x \\ +\infty & \text{otherwise,} \end{cases}$$

whose effect on the dual representation is simply to select probability measures among possibly non-normalized measures, which corresponds to selecting only those Q such that $\beta(Q) = 1$ in Proposition 19. Finally, having discussed the relaxation of convexity to GG-convexity, a natural question is whether also GG-convexity is inherited by the HG risk measure introduced defined by means of (7). Interestingly, this is not the case, as the following example shows.

Example 25 *Let X be a random variable such that $\log X \in L^1(\Omega, \mathcal{F}, P)$ and let*

$$\rho(X) = \begin{cases} \exp(\mathbb{E}[\log X]) & \text{if } P(X = 0) = 0 \\ 0 & \text{if } P(X = 0) > 0 \end{cases}.$$

Then, ρ is geometrically convex and from equation (7)

$$\rho_{\mathrm{HG}}(X) = \inf_{x \in \mathbb{R}} \{x + \exp(\mathbb{E}[\log(X - x)_+])\} = \inf_{x \in \mathbb{R}} g(x),$$

where $g(x) := x + \exp(\mathbb{E}[\log(X - x)_+])$. Letting $X = \frac{1}{2} \cdot 1_A + 2 \cdot 1_{A^c}$ with $P(A) = 1/2$, it follows that

$$g(x) = \begin{cases} x & \text{if } x \geqslant 1/2 \\ x + \sqrt{x^2 - 5/2x + 1} & \text{if } x < 1/2, \end{cases}$$

and it is not difficult to verify that $\rho_{\mathrm{HG}}(X) = 1/2$. Setting $Y = 1/X$, it follows that $X \stackrel{d}{=} Y$ so also $\rho_{\mathrm{HG}}(Y) = 1/2$. Since $XY = 1$, we have

$$1 = \rho_{\mathrm{HG}}(X^{1/2} Y^{1/2}) > [\rho_{\mathrm{HG}}(X)]^{1/2} \cdot [\rho_{\mathrm{HG}}(Y)]^{1/2} = 1/2,$$

which shows that ρ_{HG} is not geometrically convex.

5 Elicitability and the CxLS property

As is well-known, law-invariant functionals such as generalized Orlicz premia can be seen as functionals defined on sets of distribution functions. An important property that these functionals can have, is the convexity of their level sets with respect to mixtures, referred to for short in the literature as the CxLS property.

Definition 26 *Let $\mathcal{M}$ be a convex set of distribution functions of probability measures on $\mathbb{R}$. A functional $\rho\colon \mathcal{M} \to \mathbb{R}$ has the CxLS property if*

$$\rho(F) = \rho(G) = \gamma \Rightarrow \rho(\lambda F + (1-\lambda)G) = \gamma,$$

for each $\gamma \in \mathbb{R}$, $F, G \in \mathcal{M}$ and $\lambda \in (0,1)$.

The CxLS property is a necessary condition for the elicitability of a functional, which can be informally defined as the property of being the minimizer of a suitable expected loss function. The relevance of this notion has been thoroughly discussed in the financial and statistical literature; we refer to e.g., [25, 5, 45, 42].

A stream of literature has studied elicitable coherent or convex risk measures, by characterizing coherent or convex risk measures satisfying the CxLS property; see, e.g., [43, 5, 45, 19]. Most of the attention has been given to the cash-additive case, with the exception of [9], in which the case of return risk measures has been considered, leading in Theorem 2 of [9] to an axiomatization of Orlicz premia based on the CxLS property. In Theorem 28 below, we present a variant of this result that under the additional assumption of geometric convexity allows for the removal of some technical assumptions of [9]; the proof is based on a corresponding result in the additive case in [19]. First, we verify in the following lemma that generalized Orlicz premia indeed satisfy the CxLS property.

Lemma 27 *Let Φ and H_Φ be as in Definitions 6 and 7. Assume that $H_\Phi(X) = H_\Phi(Y) = \gamma \in [0, +\infty)$, with $X \sim F$ and $Y \sim G$. Then, for each $\lambda \in (0,1)$,*

$$Z \sim \lambda F + (1-\lambda)G \Rightarrow H_\Phi(Z) = \gamma.$$

As anticipated, the next theorem shows that any law-invariant and geometrically convex return risk measure with the CxLS property is necessarily a geometrically convex Orlicz premium. This result can be seen as a multiplicative version of Theorem 3.10 of [19], where, remarkably, also unbounded Orlicz functions are possible. In order to exploit the one-to-one correspondence between convex shortfall risk measures and geometrically convex return risk measures outlined in [9], we suitably restrict the domain to L^∞_{++}.

Theorem 28 *Let $\rho\colon L^\infty_{++} \to (0, +\infty)$ be a law-invariant geometrically convex return risk measure with CxLS. Then, there exists a GA-convex Orlicz function $\Phi\colon (0, +\infty) \to \mathbb{R} \cup \{+\infty\}$ such that $\rho(X) = H_\Phi(X)$.*

Since from Proposition 13, item e), it follows that a convex return risk measure is also geometrically convex, we can derive as a corollary the following similar axiomatization in the convex case.

Corollary 29 *Let $\rho\colon L^\infty_{++} \to (0,+\infty)$ be a law-invariant convex return risk measure with CxLS. Then, there exists a convex Orlicz function $\Phi\colon (0,+\infty) \to \mathbb{R} \cup \{+\infty\}$ such that $\rho(X) = H_\Phi(X)$.*

Recalling that the CxLS property is a necessary condition for elicitability, we thus find that generalized Orlicz premia naturally arise as the only elicitable geometrically convex or convex return risk measures. The study of families of loss functions consistent with Orlicz premia and their actuarial and financial applications is being pursued in a separate paper.

6 Concluding remarks

Our analysis has shown that possibly non-convex Orlicz premia constitute a viable and promising generalization of convex Orlicz premia. Our extended definition allows for a connection with various functionals, such as expectiles and other generalized quantiles, that appears to have not yet been recognized in the literature. We have established that cash-additive generalized Orlicz premia do not collapse to the mean, contrary to conventional wisdom, but instead give rise to L^p-quantiles.

Furthermore, relaxing convexity naturally leads to the broader concept of geometric convexity, which may be more appropriate for multiplicative quantities such as simple returns, as suggested in the referenced literature on return risk measures. As we have seen, geometric convexity exhibits an elegant duality theory, with the geometric mean as its fundamental building block. We believe this structure offers a beautiful parallel to the duality theory for the usual (arithmetic) convexity.

Finally, we have shown that generalized Orlicz premia naturally emerge as the broadest class of elicitable, positively homogeneous and monotonic premium principles. As such, they provide the natural framework for applying loss-based statistical methods to actuarial premium calculation. We believe that the generalized definition of Orlicz premia introduced in this paper should be adopted as the new canonical and standard one.

A Proofs

Proof of Proposition 8. (a) From the assumptions on Φ, the set $I_X := \{k > 0 \mid \mathbb{E}[\Phi(X/k)] \leqslant 1\}$ is a nonempty unbounded interval and $\operatorname{ess\,sup} X \in I_X$. Monotonicity of H_Φ follows from the monotonicity of Φ and the proof of positive homogeneity of H_Φ is standard. Since $\mathbb{E}[\Phi(1/k)] \leqslant 1$ for $k \geqslant 1$ and $\mathbb{E}[\Phi(1/k)] > 1$ for $k < 1$, we have $H_\Phi(1) = 1$.
(b) From Definition 6, it follows that $X/H_\Phi(X) \leqslant u$, which implies $u \cdot H_\Phi(X) \geqslant \operatorname{ess\,sup} X$, yielding the thesis.
(c) Let $g(k) := \mathbb{E}\left[\Phi(X/k)\right]$. Let $k \geqslant H_\Phi(X)$. If $k_n \downarrow k$, then, from left continuity of Φ, it follows that $\Phi(X/k_n) \uparrow \Phi(X/k) \leqslant \Phi(\operatorname{ess\,sup}(X)/k)$. There are two cases now. If Φ does not take the value $+\infty$, from the dominated convergence theorem

it follows that g is right-continuous. If $\Phi(x)$ takes the value $+\infty$ for some finite x, then from b) it follows that

$$\Phi\left(\frac{\operatorname{ess\,sup} X}{k}\right) \leqslant \Phi\left(\frac{\operatorname{ess\,sup} X}{\operatorname{ess\,sup} X/u}\right) = \Phi(u) < +\infty,$$

so again the right continuity of g follows from the dominated convergence theorem. Hence, $H_\Phi(X) = \inf\{k \mid g(k) \leqslant 1\} = \min\{k \mid g(k) \leqslant 1\}$.
(d) Follows from Definition 6 and (c).
(e) Under these assumptions, the function $g(k) = \mathbb{E}\left[\Phi(X/k)\right]$ introduced in the proof of (c) is continuous and strictly decreasing, from which the thesis follows.
(f) The proof is standard.
(g) Assume by contradiction that Φ is not midconvex, i.e., there exist $x_1, x_2 \in (0, +\infty)$ with $\Phi(x_1) < +\infty$, $\Phi(x_2) < +\infty$ such that

$$b := \Phi\left((x_1 + x_2)/2\right) > (\Phi(x_1) + \Phi(x_2))/2 := a.$$

We want to prove that there exist $z \in (0, +\infty)$ and $c = \Phi(z)$ such that

$$\begin{cases} \lambda c + (1-\lambda) b > 1 \\ \lambda c + (1-\lambda) a \leqslant 1 \end{cases} \tag{A.I}$$

for some $\lambda \in (0, 1)$, or equivalently that

$$c \in I_\lambda := \left(\frac{1 - b(1-\lambda)}{\lambda}, \frac{1 - a(1-\lambda)}{\lambda}\right].$$

There are three cases. If $a \leqslant 1 < b$, then $c = 1$ satisfies (A.I) for each $\lambda \in (0, 1)$. If $a < b \leqslant 1$, then $z = u_2$ and $c = \Phi(u_2)$ satisfies (A.I) since any $c > 1$ is contained in some I_λ, while if $1 \leqslant a < b$, then $z = u_1$ and $c = \Phi(u_1)$ satisfies (A.I) since any $c < 1$ is contained in some I_λ. As a consequence,

$$\lambda\Phi(z) + (1-\lambda)\Phi\left((x_1 + x_2)/2\right) > 1 \geqslant \lambda\Phi(z) + (1-\lambda)\frac{\Phi(x_1) + \Phi(x_2)}{2}. \tag{A.II}$$

Let $A, B, C \in \mathcal{F}$ be disjoint sets with $P(A) = \lambda$, $P(B) = P(C) = \frac{1-\lambda}{2}$ and let

$$\begin{aligned} X &= z1_A + x_1 1_B + x_2 1_C \\ Y &= z1_A + x_2 1_B + x_1 1_C \\ Z &= z1_A + \frac{x_1 + x_2}{2} 1_{B\cup C} = \frac{X + Y}{2}. \end{aligned}$$

From (A.II), we have $\mathbb{E}[\Phi(Z)] > 1 \Rightarrow H_\Phi(Z) > 1$ and $\mathbb{E}[\Phi(X)] \leqslant 1 \Rightarrow H_\Phi(X) \leqslant 1$, $\mathbb{E}[\Phi(Y)] \leqslant 1 \Rightarrow H_\Phi(Y) \leqslant 1$, which contradicts the convexity of H_Φ. As a consequence, Φ is midconvex where it is finite, and since it is also nondecreasing, it is convex where it is finite.
(h) Follows immediately from the definition. ∎

Proof of Theorem 9. Let $x_1 < 1$ and $x_2 > 1$. From the assumptions on Φ, there exists p such that $p\Phi(x_1) + (1-p)\Phi(x_2) = 1$, and from Proposition 8 it follows that $H_\Phi(X) = 1$, where $X = x_1 1_A + x_2 1_{A^c}$ with $P(A) = p$. From positive homogeneity and cash-additivity, it follows that $H_\Phi(cX - c + 1) = c - c + 1 = 1$ for any $c \geqslant 0$, so again from Proposition 8 it follows that $p\Phi(c(x_1-1)+1) + (1-p)\Phi(c(x_2-1)+1) = 1$, for any $c \geqslant 0$.
Let $u := x_1 - 1 < 0$ and $v := x_2 - 1 > 0$. The previous arguments have shown that if

$$pf(u) + (1-p)g(v) = 0, \tag{A.III}$$

then

$$pf(cu) + (1-p)g(cv) = 0, \text{ for any } c \geqslant 0,$$

where f and g denote the restrictions of the function $h(x) := \Phi(x+1) - 1$ to the domains $(-1, 0)$ and $(0, +\infty)$, respectively. Since from the assumptions on Φ it follows that for any $u \in (-1, 0)$ and $v \in (0, +\infty)$ it is possible to find p such that (A.III) holds, we find that for each u and v,

$$\frac{f(u)}{g(v)} = \frac{f(cu)}{g(cv)}, \text{ for any } c > 0,$$

which can be recast into a multiplicative Pexider functional equation (see, e.g., [1], Theorem 4 in Section 3.1), whose general solution is

$$\begin{cases} f(u) = -a(-u)^p \text{ if } u \leqslant 0 \\ g(v) = bv^p \text{ if } v \geqslant 0, \end{cases}$$

with $a > 0$, $b > 0$ and $p \geqslant 0$ from the assumptions on Φ, from which a) follows. To prove b), notice that convexity or concavity holds if and only if $p = 1$, and is determined by the inequality between a and b. ■

Proof of Proposition 13. Items a) and b) follow immediately from the definitions of GG- and GA-convexity. Items c) and d) follow easily from the AM-GM inequality. To show e), let $X, Y \in L^\infty_+$ and $\lambda \in (0,1)$. If $\rho(X) = 0$ or $\rho(Y) = 0$, the thesis is trivial. By using the AM-GM inequality and the monotonicity and convexity of ρ, we get

$$\begin{aligned} \rho\left(\left(\frac{X}{\rho(X)}\right)^\lambda \left(\frac{Y}{\rho(Y)}\right)^{1-\lambda}\right) &\leqslant \rho\left(\lambda\frac{X}{\rho(X)} + (1-\lambda)\frac{Y}{\rho(Y)}\right) \\ &\leqslant \lambda\rho\left(\frac{X}{\rho(X)}\right) + (1-\lambda)\rho\left(\frac{Y}{\rho(Y)}\right) = 1. \end{aligned}$$

Next, from positive homogeneity it follows that

$$\rho\left(X^\lambda Y^{1-\lambda}\right) \leqslant \rho(X)^\lambda \rho(Y)^{1-\lambda},$$

which completes the proof. ■

Proof of Proposition 14. We first prove the 'if' part. Let $X, Y \in L_+^\infty$ and $\lambda \in (0,1)$. From the GA-convexity of Φ it follows that

$$\begin{aligned}
&\mathbb{E}\left[\Phi\left(\left(\frac{X}{H_\Phi(X)}\right)^\lambda \left(\frac{Y}{H_\Phi(Y)}\right)^{1-\lambda}\right)\right] \\
&\leqslant \lambda\,\mathbb{E}\left[\Phi\left(\frac{X}{H_\Phi(X)}\right)\right] + (1-\lambda)\,\mathbb{E}\left[\Phi\left(\frac{Y}{H_\Phi(Y)}\right)\right] \leqslant 1,
\end{aligned}$$

which from Proposition 8 implies

$$H_\Phi\left(\frac{X^\lambda Y^{1-\lambda}}{H_\Phi(X)^\lambda H_\Phi(Y)^{1-\lambda}}\right) \leqslant 1,$$

which from positive homogeneity gives

$$H_\Phi\left(X^\lambda Y^{1-\lambda}\right) \leqslant H_\Phi(X)^\lambda H_\Phi(Y)^{1-\lambda}.$$

To prove the 'only if' part, we first assume by contradiction that Φ is not GA-midconvex, i.e., there exist $x_1, x_2 \geqslant 0$ with $\Phi(x_1) < +\infty$ and $\Phi(x_2) < +\infty$ such that $\Phi\left(\sqrt{x_1 x_2}\right) > (\Phi(x_1) + \Phi(x_2))/2$. Then, reasoning as in the proof of Proposition 8 item (g), there exist $z \in [0, +\infty)$ and $\lambda \in (0,1)$ such that

$$\lambda\Phi(z) + (1-\lambda)\Phi(\sqrt{x_1 x_2}) > 1 > \lambda\Phi(z) + (1-\lambda)\frac{\Phi(x_1) + \Phi(x_2)}{2}. \qquad \text{(A.IV)}$$

Take disjoint sets $A, B, C \in \mathcal{F}$ with $P(A) = \lambda$, $P(B) = P(C) = \frac{1-\lambda}{2}$ and let

$$\begin{aligned}
X &= z1_A + x_1 1_B + x_2 1_C \\
Y &= z1_A + x_2 1_B + x_1 1_C \\
Z &= z1_A + \sqrt{x_1 x_2} 1_{B \cup C} = \sqrt{XY}.
\end{aligned}$$

From (A.IV), we have $\mathbb{E}[\Phi(Z)] > 1$ and $\mathbb{E}[\Phi(X)] = \mathbb{E}[\Phi(Y)] < 1$, which contradicts with the geometric convexity of H_Φ. As a consequence, Φ is GA-midconvex and since it is also nondecreasing, the thesis follows. ∎

Proof of Corollary 15. From Theorem 9, item a), it follows that $\Phi(x) = 1 + a(x-1)_+^p - b(x-1)_-^p$ with $a > 0$, $b > 0$ and $p \geqslant 0$. From Proposition 14 it follows that Φ is GA-convex, so from Proposition 13, item b), it follows that

$$\Phi(e^x) = \begin{cases} 1 + a(e^x - 1)^p \text{ for } x \geqslant 0, \\ 1 - b(1 - e^x)^p \text{ for } x < 0, \end{cases}$$

is convex, which by a direct check implies $p = 1$. ∎

Proof of Theorem 18. The first part of the statement is easily derived from Proposition 4.3 in [33]. We can write ([33])

$$\beta(Q) = \sup\{\lambda \geqslant 0 : \lambda\,\mathbb{E}_Q[X] \leqslant \rho(X) \text{ for all } X \in L_+^\infty\}.$$

Therefore, we obtain

$$\beta(Q) = \sup\left\{\lambda \geqslant 0 : \lambda \leqslant \inf_{X \in L_+^\infty} \frac{\rho(X)}{\mathbb{E}_Q[X]}\right\}$$
$$= \inf_{X \in L_+^\infty} \frac{\rho(X)}{\mathbb{E}_Q[X]} = \inf_{\rho(X)=1} \frac{1}{\mathbb{E}_Q[X]},$$

where we use the positive homogeneity of ρ in the last equality. (When $X \equiv 0$, we set $0/0 = 1$.) Now, let us show that

$$\sup_{X \in B_\rho} \mathbb{E}_Q[X] = \sup_{\rho(X)=1} \mathbb{E}_Q[X].$$

It directly follows that $\sup_{X \in B_\rho} \mathbb{E}_Q[X] \geqslant \sup_{\rho(X)=1} \mathbb{E}_Q[X]$ since $\{X \in L_+^\infty : \rho(X) = 1\} \subseteq B_\rho$. For the other side, let us take $X \in B_\rho$ and define $\tilde{X} = \frac{X}{\rho(X)}$. Note that $\rho(\tilde{X}) = 1$. Then, we have

$$\mathbb{E}_Q[X] \leqslant \frac{\mathbb{E}_Q[X]}{\rho(X)} = \mathbb{E}_Q[\tilde{X}] \leqslant \sup_{\rho(Y)=1} \mathbb{E}_Q[Y].$$

Since this inequality is valid for any $X \in B_\rho$, if we take the supremum of the left-hand side, we obtain $\sup_{X \in B_\rho} \mathbb{E}_Q[X] \leqslant \sup_{\rho(X)=1} \mathbb{E}_Q[X]$, which completes the proof. For the second part, it follows from the proof of Proposition 4.3 in [33] that

$$\rho(X) = \sup_{Z \in H} \mathbb{E}[XZ],$$

where $H = \left\{Z \in L_+^1 : \mathbb{E}[ZY] \leqslant \rho(Y) \text{ for any } Y \in L_+^\infty\right\}$. If we take $Y = 1$, then $\mathbb{E}[Z] \leqslant 1$ for any $Z \in H$, which gives the norm-boundedness of the set H. Furthermore, H is weakly closed, since it is an intersection of weakly closed sets. Let us take a decreasing sequence $(A_n)_n \in \mathcal{F}$ of which the intersection is the empty set. For any $Z \in H$, we have $\mathbb{E}[Z 1_{A_n}] \leqslant \rho(1_{A_n})$ for every n. Therefore, by using the Lebesgue property of ρ, we have

$$\lim_{n \to +\infty} \sup_{Z \in H} \mathbb{E}[Z 1_{A_n}] \leqslant \lim_{n \to +\infty} \rho\left(1_{A_n}\right) = 0,$$

which gives that H is uniformly integrable. Because H is bounded, weakly closed and uniformly integrable, it is weakly compact as a consequence of the Dunford-Pettis theorem (see, e.g., Theorem A.67 in [22]). Therefore, the supremum is attained as a result of the Weierstrass Theorem (see, e.g., Corollary 2.35 in [2]). Suppose the supremum is attained for $\tilde{Z} \in H$. Then, the supremum is attained for $\tilde{Q}$ such that $\frac{\mathrm{d}\tilde{Q}}{\mathrm{d}P} = \frac{\tilde{Z}}{\mathbb{E}[\tilde{Z}]}$.

Now suppose that ρ is law-invariant. Hence, $X \in B_\rho$ if and only if $\tilde{X} \in B_\rho$ for all $\tilde{X} \sim X$. Using (3), we obtain

$$\beta(Q) = \left[\sup_{X \in B_\rho} \mathbb{E}_Q[X]\right]^{-1} = \left[\sup_{X \in B_\rho} \sup_{\tilde{X} \sim X} \mathbb{E}[\varphi_Q \tilde{X}]\right]^{-1} = \left[\sup_{X \in B_\rho} \int_0^1 q_X(t) q_{\varphi_Q}(t) \mathrm{d}t\right]^{-1},$$

where we used Lemma 4.60 in [22] in the last equality. This result implies that the penalty function only depends on the distribution of φ_Q. Then, using (2), we have

$$\begin{aligned}\rho(X) &= \sup_{Q\in\mathbf{P}}\{\beta(Q)\,\mathbb{E}_Q[X]\} = \sup_{Q\in\mathbf{P}}\sup_{\tilde{Q}\sim Q}\{\beta(Q)\,\mathbb{E}[\varphi_{\tilde{Q}}X]\}\\ &= \sup_{Q\in\mathbf{P}}\beta(Q)\sup_{\tilde{Q}\sim Q}\{\mathbb{E}[\varphi_{\tilde{Q}}X]\} = \sup_{Q\in\mathbf{P}}\left\{\beta(Q)\int_0^1 q_X(t)q_{\varphi_Q}(t)\,\mathrm{d}t\right\},\end{aligned}$$

where we used Lemma 4.60 in [22] in the last equality. ■

Proof of Proposition 19. The penalty function in the general dual representation (2) now takes the following specific form:

$$\frac{1}{\beta(Q)} = \sup_{X\in B_{H_\Phi}}\mathbb{E}_Q[X] = \sup_{X\in L_+^\infty}\{\mathbb{E}_Q[X] : \mathbb{E}[\Phi(X)]\leqslant 1\}.$$

Since Φ is convex, Slater's condition holds and using strong duality, we obtain

$$\begin{aligned}\frac{1}{\beta(Q)} &= \inf_{\lambda>0}\left\{\sup_{X\in L_+^\infty}\left[\mathbb{E}_Q[X]-\lambda\,\mathbb{E}[\Phi(X)]+\lambda\right]\right\}\\ &= \inf_{\lambda>0}\left\{\lambda+\sup_{X\in L_+^\infty}\mathbb{E}\left[\frac{\mathrm{d}Q}{\mathrm{d}P}X-\lambda\Phi(X)\right]\right\}\\ &= \inf_{\lambda>0}\left\{\lambda+\lambda\sup_{X\in L_+^\infty}\mathbb{E}\left[\frac{\mathrm{d}Q}{\lambda\mathrm{d}P}X-\Phi(X)\right]\right\}\\ &= \inf_{\lambda>0}\left\{\lambda+\lambda\,\mathbb{E}\left[\sup_{x\geqslant 0}\frac{\mathrm{d}Q}{\lambda\mathrm{d}P}x-\Phi(x)\right]\right\} = \inf_{\tilde{\lambda}>0}\frac{1}{\tilde{\lambda}}\,\mathbb{E}\left[1+\Psi\left(\tilde{\lambda}\frac{\mathrm{d}Q}{\mathrm{d}P}\right)\right],\end{aligned}$$

where we used Theorem 14.60 in [41] in the one but last equality, and the change of variable $\lambda = 1/\tilde{\lambda}$ in the last equality. ■

Proof of Theorem 21. From Lemma 2 in [9] it follows that $\tilde{\rho}(X) := \log(\rho(\exp(X)))$ is a convex monetary risk measure, and since ρ has the lower-bounded Fatou property it follows easily that $\tilde{\rho}$ has the Fatou property so it admits a dual representation of the form (see, e.g., [22, 18])

$$\tilde{\rho}(X) = \sup_{Q\in\mathbf{P}}\{\mathbb{E}_Q[X]-\tilde{\alpha}(Q)\}, \tag{A.V}$$

where $\tilde{\alpha}(Q) = \sup_{X\in A_{\tilde{\rho}}}\mathbb{E}_Q[X]$, and $A_{\tilde{\rho}}$ is the acceptance set of $\tilde{\rho}$. Since $\rho(X) = \exp(\tilde{\rho}(\log(X)))$, it follows that

$$\begin{aligned}\rho(X) &= \exp\left(\sup_{Q\in\mathbf{P}}\{\mathbb{E}_Q[\log X]-\tilde{\alpha}(Q)\}\right) = \sup_{Q\in\mathbf{P}}\{\exp\left(\mathbb{E}_Q[\log X]-\tilde{\alpha}(Q)\right)\}\\ &= \sup_{Q\in\mathbf{P}}\{\alpha(Q)\exp(\mathbb{E}_Q[\log X])\},\end{aligned}$$

where $\alpha\colon \mathbf{P} \to [0,1]$ is given by

$$\alpha(Q) = \exp(-\tilde{\alpha}(Q)) = \frac{1}{\exp\left(\sup_{Y\in A_{\tilde{\rho}}} \mathbb{E}_Q[Y]\right)} = \frac{1}{\exp\left(\sup_{X\in B_\rho} \mathbb{E}_Q[\log X]\right)},$$

where we set $Y = \log(X)$ in the last equation. From $\rho(1) = 1$, it follows that $\sup_{Q\in\mathbf{P}} \alpha(Q) = 1$. If ρ satisfies the lower-bounded Lebesgue property, then ρ has the Lebesgue property, and by Theorem 4.22 and Exercise 4.2.2 in [22], the supremum in (A.V) is attained, so the supremum in (5) is also attained.

Now suppose that ρ is law-invariant. Hence, $X \in B_\rho$ if and only if $\tilde{X} \in B_\rho$ for all $\tilde{X} \sim X$. Using (6), we obtain

$$\begin{aligned}
\alpha(Q) &= \left[\sup_{X\in B_\rho} \exp(\mathbb{E}_Q[\log X])\right]^{-1} = \left[\exp\left(\sup_{X\in B_\rho} \sup_{\tilde{X}\sim X} \mathbb{E}[\varphi_Q \log \tilde{X}]\right)\right]^{-1} \\
&= \left[\sup_{X\in B_\rho} \exp\left(\int_0^1 q_{\log X}(t) q_{\varphi_Q}(t) \mathrm{d}t\right)\right]^{-1},
\end{aligned}$$

where we used Lemma 4.60 in [22] in the last equality. This result implies that the penalty function only depends on the distribution of φ_Q. Then, using (5), we have

$$\begin{aligned}
\rho(X) &= \sup_{Q\in\mathbf{P}} \{\alpha(Q) \exp\left(\mathbb{E}_Q[\log X]\right)\} = \exp\left(\sup_{Q\in\mathbf{P}} \sup_{\tilde{Q}\sim Q} \{\alpha(Q)\, \mathbb{E}[\varphi_{\tilde{Q}} \log X]\}\right) \\
&= \exp\left(\sup_{Q\in\mathbf{P}} \alpha(Q) \sup_{\tilde{Q}\sim Q} \mathbb{E}[\varphi_{\tilde{Q}} \log X]\right) = \sup_{Q\in\mathbf{P}} \left\{\exp\left(\alpha(Q) \int_0^1 q_{\log X}(t) q_{\varphi_Q}(t)\, \mathrm{d}t\right)\right\},
\end{aligned}$$

where we used Lemma 4.60 in [22] in the last equality. ■

Proof of Proposition 24. We start from the well-known variational formula linking the exponential certainty equivalent and relative entropy (see, e.g., [20])

$$\log \mathbb{E}_Q[\exp(Y)] = \sup_{R\ll Q} \{\mathbb{E}_R[Y] - \mathrm{H}(R,Q)\}, \tag{A.VI}$$

where $\mathrm{H}(R,Q)$ is the relative entropy as in Definition 23. Letting $X = \exp(Y)$ and exponentiating both sides of (A.VI), we obtain

$$\mathbb{E}_Q[X] = \sup_{R\ll Q} \{\alpha(R) \exp\left(\mathbb{E}_R[\log X]\right)\}, \tag{A.VII}$$

where

$$\alpha(R) = \exp\left(-\,\mathrm{H}(R,Q)\right).$$

Now note that $\alpha(R) = 0$ when R is not absolutely continuous with respect to Q. Using this fact, we can rewrite expression (A.VII) for $Q \in \mathbf{P}$, as follows:

$$\begin{aligned}
\mathbb{E}_Q[X] &= \sup_{R\ll Q} \{\alpha(R) \exp\left(\mathbb{E}_R[\log(X)]\right)\} \\
&= \sup_{R\in\mathbf{P}} \{\alpha(R) \exp\left(\mathbb{E}_R[\log(X)]\right)\}, \qquad \text{(A.VIII)}
\end{aligned}$$

since we take a supremum of nonnegative numbers, $\alpha(R) = 0$ when $R \notin \mathbf{Q}$ and $\mathbf{Q} \subset \mathbf{P}$, where $\mathbf{Q}$ denotes the set of probability measures absolutely continuous with respect to Q. Substituting the expression for $\mathbb{E}_Q[X]$ derived in (A.VIII) in (2), we get

$$\begin{aligned}
\rho(X) &= \sup_{Q\in\mathbf{P}}\{\beta(Q)\,\mathbb{E}_Q[X]\} = \sup_{Q\in\mathbf{P}}\left\{\beta(Q)\sup_{R\ll Q}\{\alpha(R)\exp\left(\mathbb{E}_R[\log X]\right)\}\right\} \\
&= \sup_{Q\in\mathbf{P}}\left\{\beta(Q)\sup_{R\in\mathbf{P}}\{\alpha(R)\exp\left(\mathbb{E}_R[\log X]\right)\}\right\} \\
&= \sup_{R\in\mathbf{P}}\sup_{Q\in\mathbf{P}}\{\beta(Q)\{\exp\left(-\mathrm{H}(R,Q)\right)\exp\left(\mathbb{E}_R[\log X]\right)\}\} \\
&= \sup_{R\in\mathbf{P}}\{c(R)\exp\left(\mathbb{E}_R[\log X]\right)\},
\end{aligned}$$

where

$$c(R) = \sup_{Q\in\mathbf{P}}\beta(Q)\exp\left(-\mathrm{H}(R,Q)\right).$$

■

Proof of Lemma 27. Let us first consider the case $\Phi(0) = -\infty$ and either $P(X = 0) > 0$ or $P(Y = 0) > 0$. This implies that $P(Z = 0) > 0$, so $H_\Phi(Z) = 0$. Otherwise, from the definition of the Orlicz premium there must exist two sequences $h_n, k_n \downarrow \gamma$ such that

$$\mathbb{E}[\Phi(X/h_n)] \leqslant 1 \text{ for each } h_n, \qquad \mathbb{E}[\Phi(Y/k_n)] \leqslant 1 \text{ for each } k_n,$$

and $\gamma' < \gamma$ implies $\mathbb{E}[\Phi(X/\gamma')] > 1$ and $\mathbb{E}[\Phi(Y/\gamma')] > 1$. Letting $\ell_n := \max(h_n, k_n)$ it follows that $\ell_n \downarrow \gamma$,

$$\mathbb{E}[\Phi(Z/\ell_n)] \leqslant \lambda\,\mathbb{E}[\Phi(X/h_n)] + (1-\lambda)\,\mathbb{E}[\Phi(Y/k_n)] \leqslant 1$$

and $\mathbb{E}[\Phi(Z/\gamma')] = \lambda\,\mathbb{E}[\Phi(X/\gamma')] + (1-\lambda)\,\mathbb{E}[\Phi(Y/\gamma')] > 1$, so $H_\Phi(Z) = \gamma$. ■

Proof of Theorem 28. For ease of reference, we recall in the following proposition some of the results given in Theorem 3.10 and Lemma 3.7 of [19], given without proof and adapted to the sign conventions of the present paper.

Proposition 30 *Let $\rho\colon L^\infty \to \mathbb{R}$ be a convex, law-invariant, monotone and cash-additive risk measure with CxLS. Then, there exists a nondecreasing, convex and left-continuous $\varphi\colon \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ satisfying $\varphi(0) = 0$ such that $\rho(X) \leqslant 0 \iff \mathbb{E}[\varphi(X)] \leqslant 0$.*

From the hypotheses and Lemma 2 in [9], it follows that $\rho = \log \circ \rho \circ \exp$ is a convex law-invariant monetary risk measure, given at the level of distributions by $\rho(F) = \log\left(\rho(F \circ \log)\right)$. If $\rho(F) = \rho(G) = \gamma$, then $\rho(F \circ \log) = \rho(G \circ \log) = \exp(\gamma)$, and since ρ has the CxLS property, $\rho(\lambda(F \circ \log) + (1-\lambda)(G \circ \log)) = \exp(\gamma)$, so $\rho((\lambda F + (1-\lambda)G) \circ \log) = \exp(\gamma)$, implying that also ρ has the CxLS property, hence it satisfies all the assumptions of Proposition 30.

From Proposition 30, it follows that there exists a nondecreasing convex and left-continuous $\varphi\colon \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ satisfying $\varphi(0) = 0$ such that $\rho(X) \leqslant 0$ if and only if $\mathbb{E}[\varphi(X)] \leqslant 0$. From convexity, it follows that φ is continuous where it is finite. Letting $\Phi(x) := 1 + \varphi(\log(x))$, it follows that Φ is an Orlicz function in the sense of Definition 6, and

$$\begin{aligned}\rho(X/k) \leqslant 1 &\iff \rho(\log(X/k)) \leqslant 0 \iff \mathbb{E}[\varphi(\log(X/k))] \leqslant 0 \\ &\iff \mathbb{E}[\Phi(X/k)] \leqslant 1,\end{aligned}$$

so $\rho(X) = H_\Phi(X)$. Finally, from the convexity of φ, for each $x, y > 0$ and $\lambda \in (0, 1)$,

$$\begin{aligned}\Phi(x^\lambda y^{1-\lambda}) &= 1 + \varphi(\log(x^\lambda y^{1-\lambda}) = 1 + \varphi(\lambda \log(x) + (1-\lambda)\log(y)) \\ &\leqslant 1 + \lambda\varphi(\log(x)) + (1-\lambda)\varphi(\log(y)) = \lambda\Phi(x) + (1-\lambda)\Phi(y),\end{aligned}$$

which shows the GA-convexity of Φ. ■

Proof of Corollary 29. Since a positively homogeneous, monotone and convex functional defined on L^∞_{++} is geometrically convex from Proposition 13, it follows from Theorem 28 that there exists a GA-convex Orlicz function $\Phi\colon (0, +\infty) \to \mathbb{R} \cup \{+\infty\}$ such that $\rho(X) = H_\Phi(X)$. Since from Proposition 8 H_Φ is convex only if Φ is convex, the thesis follows. ■

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