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DICE simplified

**Masako Ikefuji · Roger J. A. Laeven ·
Jan R. Magnus · Chris Muris**

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Abstract We analyze Nordhaus' DICE model and show that the temperature and CO_2 equations are needlessly complicated and can be simplified without loss of essence. In addition, we argue that the damage function can be altered in such a way that it lends itself to experiments involving extreme risk. We conclude that, within the philosophy of the DICE model, significant simplifications can be made which make the model more transparent, more robust, and easier to apply.

Keywords DICE model · Climate change · Integrated assessment model · Optimal policy · Damage function

Mathematics Subject Classification (2010) 93A30 · 91B76

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Masako Ikefuji
Faculty of Humanities and Social Sciences, University of Tsukuba
and ISER, Osaka University, Japan
E-mail: ikefuki.masako.gn@u.tsukuba.ac.jp

Roger J. A. Laeven
Amsterdam School of Economics, University of Amsterdam, CentER,
and EURANDOM, The Netherlands
E-mail: R.J.A.Laeven@uva.nl

Jan R. Magnus
Department of Econometrics and Data Science, Vrije Universiteit Amsterdam,
and Tinbergen Institute, The Netherlands
E-mail: jan@janmagnus.nl

Chris Muris
Department of Economics, McMaster University, Canada
E-mail: muerisc@mcmaster.ca

Entia non sunt multiplicanda praeter necessitatem
William of Ockham, 1285(?)–1347

1 Introduction

This paper provides a critical assessment of the 2016 DICE (Dynamic Integrated model of Climate and the Economy) model originally developed by Nordhaus [17], but since then continuously updated and altered (see [22]). Integrated assessment models (IAMs) are being used extensively for the analysis of climate change policy and DICE has played an important part in projecting greenhouse gas emissions and temperature under various social and economic scenarios.

The assumptions and functional forms used in DICE have been under considerable scrutiny and criticism. Among others, [23, 24, 8, 9] argue that IAMs should be made simpler and more transparent, and an editorial in *Nature Climate Change* [6] supports this view. DICE is complex, because it employs an equation for radiative forcing, two equations (a two-box model) for the climate system, and a three-reservoir model for the carbon cycle. Some of the other IAMs (for example, the FUND and PAGE models) are somewhat less complex, as they employ a single-equation climate model (see [3]).

The purpose of this paper is to show that the temperature and CO_2 equations in DICE are needlessly complicated and can be much simplified. The technical reason why this simplification is possible lies in the fact that the matrix connecting the two dynamic equations describing temperature in the DICE model has one eigenvalue close to one, and that the matrix connecting the three equations describing CO_2 has one exact and one approximate unit eigenvalue. We derive the equivalent equations after differencing out the auxiliary variables and we provide simplifications.

In response to criticism that IAMs give the impression of being 'black boxes' ([6]), one approach is to propose significant changes to IAMs such as DICE, leading to simpler IAMs with an analytical formula for the optimal carbon price; see [10, 29, 25, 30, 5, 31]. Among studies employing closed-form IAMs, the specifications of the climate system and the carbon cycle vary. Golosov et al. [10] specify a two-and-a-half-box carbon system consisting of a permanent component (about 20% of carbon) and a transient component. The climate system is omitted. Under assumptions such as logarithmic utility, Cobb-Douglas technology, constant saving rate, and full depreciation of capital, they derive the optimal-tax formula analytically. Rezai and Van der Ploeg [25] employ a similar two-box carbon system. They allow for a lag between temperature and atmospheric carbon in addition to more general functional forms than those assumed by [10], and they derive a simple rule for the optimal carbon price.

The omission of the climate system in [10] is justified by recent findings in climate science that the climate response to a CO_2 emission is nearly instantaneous and remains almost constant over time; see [27, 26]. Based on the same findings, Dietz and Venmans [5] assume that the global mean temperature is linearly proportional to cumulative CO_2 emissions, in contrast to the large thermal inertia of the climate system assumed in DICE and [16]. These differences may lead to different optimal transition paths of temperature. Thus, the optimal carbon price follows Hotelling's rule in [5], whereas it grows more slowly in [16].

Our approach is to stay as close as possible to Nordhaus' DICE model, but to simplify it using Ockham's razor quoted above ('more things should not be used than are necessary'). A complex model is not necessarily better than a simple model. Leaving things out is arguably more difficult and more important than putting things in. Many statisticians believe that a more complex model will reduce the bias and increase the variance, but this is only half true. A more complex model does indeed increase the variance, but it does not necessarily reduce the bias (see [4]). Hence, simplicity matters given the large uncertainties in the exogenous variables (such as population and technical knowledge) and the parameters.

We shall not propose a completely different climate system nor do we examine climate sensitivity in the DICE model. Our purpose is more modest. We shall show that the temperature and CO_2 equations in the DICE model are needlessly complicated and that they can be much simplified. We also argue that the specification of the damage function can be altered in such a way that it lends itself to experiments involving extreme risk. Finally, we briefly discuss the assumption in DICE that the abatement fraction for CO_2 is allowed to become larger than one, which implies that emissions can become negative.

In Sect. 2 we present the Nordhaus DICE 2016R model. In Sect. 3 and 4 we discuss and simplify the DICE equations for temperature and CO_2 concentration. In Sect. 5 we provide an alternative to the DICE damage function. Sect. 6 presents and optimizes the S-DICE (simplified DICE) model and concludes.

2 Nordhaus' DICE 2016R model

The following equations are the equations from the beta version of DICE-2016R [20,21], a version with the identification DICE-2016R-091916ap.gms. A number of equations are redundant and have been deleted. A new variable ω_t has been introduced, some equations have been combined, and the equations have been reordered; see [14] for the details. Still, this is *precisely* the same model as Nordhaus' 2016R model.

Everybody works. In period t, the labor force L_t together with the capital stock K_t generate GDP Y_t through a Cobb-Douglas production function

$$Y_t = A_t K_t^{\gamma} L_t^{1-\gamma} \qquad (0 < \gamma < 1), \tag{1}$$

where A_t represents technological efficiency and γ is the elasticity of capital. Capital is accumulated through

$$K_{t+1} = (1-\delta)K_t + I_t \qquad (0 < \delta < 1), \tag{2}$$

where I_t denotes investment and δ is the depreciation rate of capital.

Carbon dioxide (CO_2) emissions consist of industrial and non-industrial ('land-use') emissions. We denote the latter type by E_t^0 and consider it to be exogenous to our model. Total CO_2 emissions E_t are then given by

$$E_t = \sigma_t(1-\mu_t)Y_t + E_t^0, \tag{3}$$

where σ_t denotes the emissions-to-output ratio for CO_2 and μ_t is the abatement fraction for CO_2. The associated CO_2 concentration increase M_t in the atmosphere

(GtC from 1750) accumulates through

$$M_{t+1} = (1 - b_0)M_t + b_1 X_{1,t} + E_t, \tag{4a}$$
$$X_{1,t+1} = b_0 M_t + (1 - b_1 - b_3)X_{1,t} + b_2 X_{2,t}, \tag{4b}$$
$$X_{2,t+1} = b_3 X_{1,t} + (1 - b_2)X_{2,t}, \tag{4c}$$

where $X_{1,t}$ and $X_{2,t}$ are auxiliary variables representing CO_2 concentration increases in shallow and lower oceans, respectively, also measured in GtC from 1750.

Temperature increase H_t (degrees Celsius from 1900) develops according to

$$H_{t+1} = (1 - a_0)H_t + a_1 \log(M_{t+1}) + a_2 Z_t + F_{t+1}, \tag{5a}$$
$$Z_{t+1} = (1 - a_3)Z_t + a_3 H_t, \tag{5b}$$

where Z_t is an auxiliary variable representing the temperature increase of the lower oceans, also measured in degrees Celsius from 1900, and F_{t+1} is exogenous radiative forcing.

In each period t, the fraction of GDP not spent on abatement or 'damage' is either consumed (C_t) or invested (I_t) along the budget constraint

$$\left(1 - \omega_t - \xi H_t^2\right) Y_t = C_t + I_t. \tag{6}$$

A fraction ω_t of Y_t is spent on abatement, and we specify the abatement cost fraction as

$$\omega_t = \psi_t \mu_t^\theta \qquad (\theta > 1). \tag{7}$$

When μ_t increases then so does ω_t, and a larger fraction of GDP will be spent on abatement.

Damage is represented by a fraction ξH_t^2 of Y_t and it depends only on temperature. The optimal temperature is $H_t = 0$, the temperature in 1900. Deviations from the optimal temperature cause damage. For very high and very low temperatures the fraction becomes large, but (given the value of ξ) it will still be a fraction between zero and one, unless in truly catastrophic cases.

As in [20, 21] one period is five years. Period 1 refers to the time interval 2015–2019, period 2 to 2020–2024, and so on. Stock variables are measured at the beginning of the period; for example, K_1 denotes capital in the year 2015. We choose the exogenous variables such that $L_t > 0$, $A_t > 0$, $E_t^0 > 0$, $\sigma_t > 0$, and $0 < \psi_t < 1$. The policy variables must satisfy

$$C_t \geq 0, \quad I_t \geq 0, \quad \mu_t \geq 0. \tag{8}$$

Nordhaus [21, 22] allows negative-emission technologies by setting an upper bound on μ_t of 1.2 (rather than 1.0) from period 30 onwards (year 2160), which implies that emissions can become negative by (3), and in fact this upper bound is reached in the DICE output from period 46 onwards (year 2240). The idea of negative emissions is controversial. Anderson and Peters [2] state that negative-emission technologies are unjust and a high-stake gamble, while a recent editorial in *Nature* ([7]) discusses the enormous effort required to carry out such technologies — an effort which would lead to a deterioration of the environment.

Given a utility function U we define welfare in period t as

$$W_t = L_t \, U(C_t/L_t). \tag{9}$$

The policy maker has a finite horizon and maximizes total discounted welfare

$$W = \sum_{t=1}^{T} \frac{W_t}{(1+\rho)^t} \qquad (0 < \rho < 1), \tag{10}$$

where ρ denotes the discount rate and $T = 100$ (500 years). Letting x denote per capita consumption, the utility function $U(x)$ is assumed to be defined and strictly concave for all $x > 0$. There are many such functions, but a popular choice is

$$U(x) = \frac{x^{1-\alpha} - 1}{1 - \alpha} \qquad (\alpha > 0), \tag{11}$$

where α denotes the elasticity of marginal utility of consumption. This is the so-called *power* function. Many authors, including Nordhaus, select this function. In earlier versions of the DICE model, Nordhaus [18] chooses $\alpha = 2$ in which case $U(x) = 1 - 1/x$. Also popular is $\alpha = 1$ in which case $U(x) = \log(x)$; see [15,28]. In the 2016 version of the DICE model $\alpha = 1.45$.

3 Temperature

The DICE model thus consists of the seven equations (1)–(7). Four of these, equations (1)–(3) and (7), are not controversial. In the next three sections we shall discuss the CO_2 equation (4), the temperature equation (5), and the budget constraint (6).

We start with the temperature equations in (5), which we now write in matrix form as

$$x_{t+1} = Ax_t + a_{t+1}, \tag{12}$$

where

$$A = \begin{pmatrix} 1 - a_0 & a_2 \\ a_3 & 1 - a_3 \end{pmatrix}$$

and

$$x_t = \begin{pmatrix} H_t \\ Z_t \end{pmatrix}, \qquad a_t = \begin{pmatrix} a_1 \log(M_t) + F_t \\ 0 \end{pmatrix}.$$

The matrix A has two eigenvalues given by

$$1 - \frac{1 - \eta_1}{2} \pm \frac{1}{2}\sqrt{(1 - \eta_1)^2 - 4\eta_2},$$

where $\eta_1 = 1 - a_0 - a_3 = 0.8468$ and $\eta_2 = (a_0 - a_2)a_3 = 0.0030$, so that the eigenvalues are 0.9771 and 0.8697, respectively. The largest eigenvalue is thus close to one and it would be equal to one if (and only if) $\eta_2 = 0$.

We can 'difference out' the auxiliary variable Z and this gives

$$\begin{aligned} H_{t+1} &= (1 + \eta_1)H_t - (\eta_1 + \eta_2)H_{t-1} \\ &\quad + \eta_3 \log(M_{t+1}) - (\eta_3 - \eta_4)\log(M_t) + \eta_{0t}, \end{aligned} \tag{13}$$

where $\eta_3 = a_1 = 0.5338$, $\eta_4 = a_1 a_3 = 0.0133$, and $\eta_{0t} = F_{t+1} - (1 - a_3)F_t$. Eq. (13) does not contain Z but, compared to (5a), it contains an additional lag in both H and $\log(M)$. Note that (13) is not invariant to scaling in M.

Letting Δ be the (backward) difference operator defined by $\Delta x_{t+1} = x_{t+1} - x_t$, we can write (13) alternatively as

$$\Delta H_{t+1} = \eta_1 \Delta H_t + \eta_3 \Delta \log(M_{t+1}) + \eta^*_{0t},$$

where $\eta^*_{0t} = -\eta_2 H_{t-1} + \eta_4 \log(M_t) + \eta_{0t}$. This equation in first differences can in turn be integrated to

$$H_{t+1} = \eta_0 + \eta_1 H_t + \eta_3 \log(M_{t+1}) + \sum_{j=1}^{t} \eta^*_{0j}, \tag{14}$$

where $\eta_0 = -3.3291$ is an integration constant. Notice that H_{t+1} in (14) depends on H_t but that the effect of H_{t-1} (through η^*_{0t}) is negligible, which is another way of saying that the largest eigenvalue of the matrix A in (12) is close to one. Both are caused by the fact that η_2 is small. We emphasize that (12), (13), and (14) are *equivalent* descriptions of the DICE temperature equations. No approximation has yet taken place.

Given the DICE parameter values, in particular the fact that η_2 and η_4 are small, the partial sums $\sum_j \eta^*_{0j}$ are well approximated by a linear trend with slope 0.025. This implies that if we run a regression on the equation

$$H_{t+1} = \eta^*_0 + \eta^*_1 H_t + \eta^*_2 \log(M_{t+1}) + \eta^*_3 t \tag{15}$$

we will get a good fit. If we leave out the linear trend, then the estimate of η^*_1 increases somewhat and the estimate of η^*_2 decreases somewhat.

Table 1 Simplified temperature equations. In (a), we report estimated coefficients and standard errors from a regression based on (15). Results without a time trend are in (b).

	constant	H_t	$\log(M_{t+1})$	*trend*/10	s	$\max \lvert Q_t \rvert$
(a)	−3.6772	0.8707	0.5826	0.0064	0.0032	0.43%
	(0.0511)	(0.0014)	(0.0076)	(0.0003)		
(b)	−2.8672	0.8954	0.4622	—	0.0068	1.70%
	(0.0561)	(0.0012)	(0.0082)			

More precisely, we obtain the results in Table 1, where we note that in all regressions, figures, and numerical experiments that follow, H_t (and similarly M_t and other variables) take the optimal values as obtained from the General Algebraic Modeling System (GAMS) routine which optimizes welfare (10) in the DICE system.

Under (a) we report the estimated coefficients and standard errors from a regression of H_{t+1} on a constant, H_t, $\log(M_{t+1})$, and a time trend, as in (15). The fit is very good. In particular, letting e denote the vector of residuals and $\hat{H}_t$ the predicted value of H_t from the regression, and defining the regression variance s^2 and the relative deviations Q_t as

$$s^2 = e'e/(n-k), \qquad Q_t = 100(\hat{H}_t - H_t)/H_t, \tag{16}$$

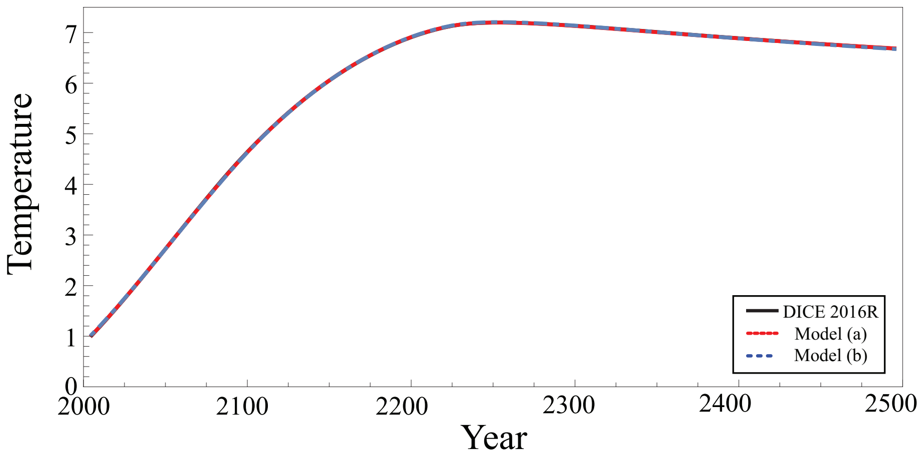

Figure 1 Temperature — time path for DICE 2016R and two simplified models. Model (a) is based on (15). Model (b) is based on (15) without a time trend.

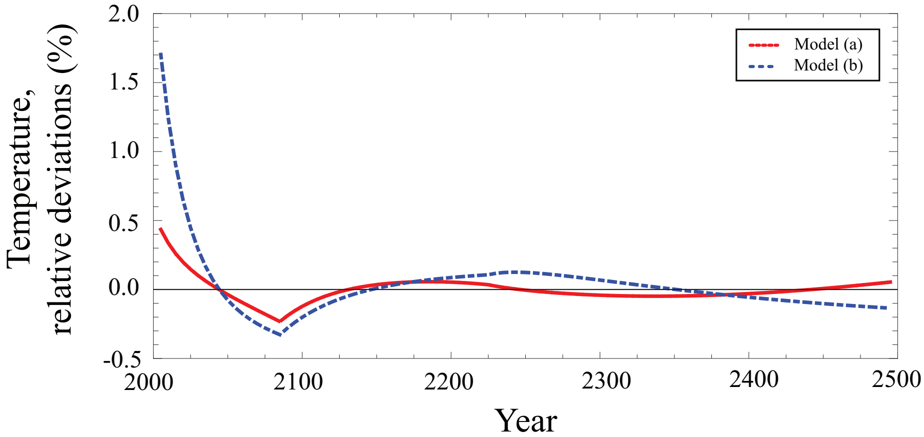

Figure 2 Temperature — Deviations (%) of two simplified models relative to DICE 2016R. Model (a) is based on (15). Model (b) is based on (15) without a time trend.

we find that $s = 0.0032$ and $\max_t |Q_t| = 0.43$ with $n = 99$ and $k = 4$. This shows that for a temperature increase of, say, 3 degrees Celsius the maximum error will be 0.013 degrees.

If we leave out the linear trend we obtain (b) which is almost as good, except that the error in the first few periods is somewhat higher. This is illustrated in Figures 1 and 2. In Figure 1 the time paths of temperature in DICE and the two models (a) and (b) are indistinguishable, reaching a maximum of 7.2 in 2270. The relative deviations Q_t are graphed in Figure 2. They are all below 0.5% except the first four periods in Model (b). Even though the estimated coefficient on the time trend is 'significant' it is not important, and the fit is essentially the same.

Summarizing, the simplified equation (15) provides a good approximation because (a) the coefficients in (13) correspond approximately to a first-order difference equation; and (b) the omitted variable is essentially constant. The second approximation (without trend) is almost as good as the approximation in (15), and suffices for practical applications.

4 CO_2 concentration

Next we consider the CO_2 equations in (4), which we also write in matrix form as

$$x_{t+1} = Ax_t + a_t, \tag{17}$$

where now

$$A = \begin{pmatrix} 1-b_0 & b_1 & 0 \\ b_0 & 1-b_1-b_3 & b_2 \\ 0 & b_3 & 1-b_2 \end{pmatrix}$$

and

$$x_t = \begin{pmatrix} M_t \\ X_{1,t} \\ X_{2,t} \end{pmatrix}, \qquad a_t = \begin{pmatrix} E_t \\ 0 \\ 0 \end{pmatrix}.$$

One of the three eigenvalues of A equals one, and the two remaining eigenvalues are given by

$$1 - \frac{1-\phi_1}{2} \pm \frac{1}{2}\sqrt{(1-\phi_1)^2 - 4\phi_2},$$

where $\phi_1 = 1-b_0-b_1-b_2-b_3 = 0.675535$ and $\phi_2 = b_0b_2+b_0b_3+b_1b_2 = 0.001303$, so that the two remaining eigenvalues take the values 0.995933 and 0.679602, respectively. The largest eigenvalue is thus equal to one and the next eigenvalue is close to one; it would be equal to one if (and only if) $\phi_2 = 0$. This suggests that we should difference not once (as in the previous section) but twice, and this is precisely what we shall do.

As in the previous section we can 'difference out' the auxiliary variables X_1 and X_2, and this gives

$$\begin{aligned} M_{t+1} = (\phi_1+2)M_t &- (1+2\phi_1+\phi_2)M_{t-1} \\ &+ (\phi_1+\phi_2)M_{t-2} + E_t^*, \end{aligned} \tag{18}$$

where

$$E_t^* = E_t - (1+\lambda_1)E_{t-1} + (\lambda_1+\lambda_2)E_{t-2}.$$

This equation does not contain X_1 and X_2 but it contains two additional lags in both M and E. We can write (18) alternatively as

$$\Delta M_{t+1} = (\phi_1+1)\Delta M_t - (\phi_1+\phi_2)\Delta M_{t-1} + E_t^*,$$

where we notice that there is no remainder term in M_{t-2} because the largest eigenvalue of A equals one exactly given the DICE parameters. This leads to

$$M_{t+1} = \phi_0 + (\phi_1+1)M_t - (\phi_1+\phi_2)M_{t-1} + E_t^{**}, \tag{19}$$

where $E_t^{**} = \sum_{j=1}^{t} E_j^*$ and $\phi_0 = 0.8761$ is an integration constant. This, in turn, can be written as

$$\Delta M_{t+1} = \phi_0 + \phi_1\Delta M_t - \phi_2 M_{t-1} + E_t^{**},$$

so that

$$M_{t+1} = \phi_{00} + \phi_0 t + \phi_1 M_t - \phi_2 \sum_{j=1}^{t-1} M_j + \sum_{j=1}^{t} w_{tj} E_j, \tag{20}$$

where $\phi_{00} = 263.2837$ is an integration constant and

$$\begin{aligned}\sum_{j=1}^{t} w_{tj} E_j &= \sum_{j=1}^{t} E_j^{**} = \sum_{j=1}^{t} \sum_{i=1}^{j} E_i^* = \sum_{j=1}^{t} (t-j+1) E_j^* \\ &= E_t + (1-\lambda_1) \sum_{j=1}^{t-1} E_j + \lambda_2 \sum_{j=1}^{t-2} (t-j-1) E_j,\end{aligned}$$

so that $w_{tt} = 1$ and

$$w_{tj} = 1 - \lambda_1 + (t-j-1)\lambda_2 \qquad (j = 1, \ldots, t-1).$$

The DICE weights w_{tj} are thus slightly *increasing* rather than decreasing, which is a little awkward. Notice that equations (17)–(20) are *equivalent* descriptions of the DICE CO_2 equations. No approximation has yet taken place.

Since $\phi_2 = 0.0013$ and $\lambda_2 = 0.0003$ are close to zero, (20) will be well approximated by

$$M_{t+1} \approx \phi_{00} + \phi_0 t + \phi_1 M_t + E_t + (1-\lambda_1) \sum_{j=1}^{t-1} E_j.$$

In fact we will run regressions on the equation

$$M_{t+1} = \phi_0^* + \phi_1^* M_t + \phi_2^* E_t + \phi_3^* t \tag{21}$$

and simplifications thereof.

Table 2 Simplified CO_2 equations. Panel (a) reports estimated coefficients and standard errors for an unrestricted regression based on (21). Panel (b) restricts the time trend to be zero. Panel (c) additionally restricts the constant term to be zero. Panel (d) additionally restricts the coefficient of E_t to be one. The statistics s and Q_t are defined in (16).

	constant	M_t	E_t	*trend*	s	$\max \lvert Q_t \rvert$
(a)	16.27	0.9900	0.6166	0.0317	1.52	0.35%
	(1.08)	(0.0004)	(0.0067)	(0.0098)		
(b)	17.94	0.9902	0.6001	—	1.59	0.39%
	(0.99)	(0.0004)	(0.0046)			
(c)	—	0.9975	0.6549	—	3.33	0.92%
		(0.0002)	(0.0073)			
(d)	—	0.9942	1.0000	—	16.35	1.66%
		(0.0007)	(—)			

This leads to the results in Table 2. Under (a) we regress M_{t+1} on all four variables, under (b) we delete the trend, under (c) we also delete the constant term, and under (d) we restrict the coefficient of E_t to be one. The last model is

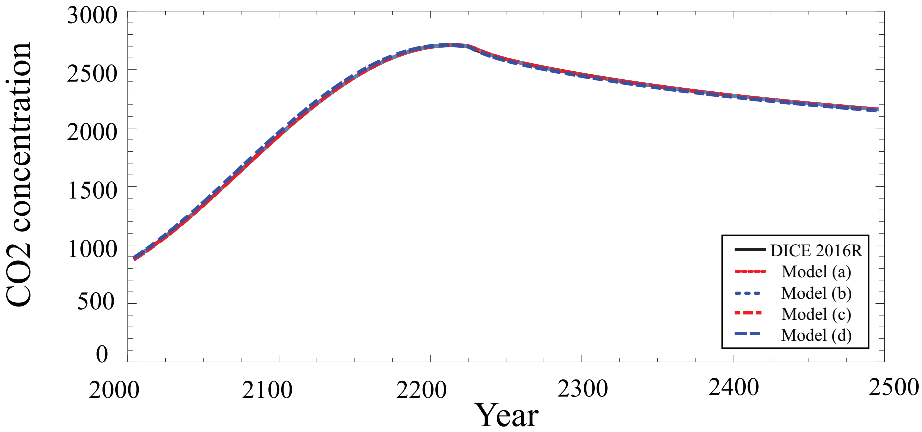

Figure 3 CO_2 concentration — time path for DICE 2016R and four simplified models. Model (a) is based on (21). Model (b) is based on (21) without a time trend. Model (c) is based on (21) without both a time trend and a constant term. Model (d) additionally restricts the coefficient of E_t to be one.

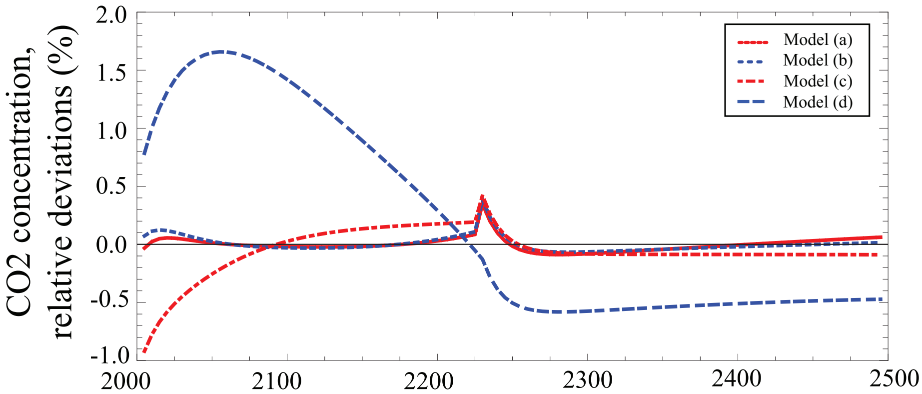

Figure 4 CO_2 concentration — Deviations (%) of four simplified models relative to DICE 2016R. Model (a) is based on (21). Model (b) is based on (21) without a time trend. Model (c) is based on (21) without both a time trend and a constant term. Model (d) additionally restricts the coefficient of E_t to be one.

the simplest and mirrors capital accumulation in (2). The fit is very good in all cases, as can be seen from the values of s and $\max |Q_t|$, and also from Figures 3 and 4.

In Figure 3 the time paths of CO_2 of DICE and the four models (a)–(d) are indistinguishable, reaching a maximum of 2707 in 2230. The relative deviations Q_t are graphed in Figure 4. In Models (a) and (b) the relative deviations are all below 0.4% in absolute value. In Model (c) the relative deviations are all below 0.6% in absolute value, except in the first three periods. In Model (d) the relative deviations are larger than 0.6% up to period 36 (year 2190) and smaller than 0.6% afterwards, with a maximum of 1.7% in period 12 (year 2070) where $M_t = 1402$ (the DICE output) and $\hat{M}_t = 1425$ (the predicted value of M_t from the regression).

The simplified equation (21) thus provides an excellent approximation to the DICE results because the coefficients in (18) correspond approximately to a second-order difference equation. Model (b) is possibly the preferred approximation although the simplest Model (d) will suffice for most practical applications.

5 Damage and abatement

The damage-abatement function in DICE specifies two fractions, ω_t (abatement) and ξH_t^2 (damage), of Y_t which reduce Y_t so that less money is available for investment and consumption along the budget constraint. In DICE this fraction is specified as

$$1 - \omega_t - \xi H_t^2.$$

For very high and very low temperatures the fraction becomes large, but (given the value of ξ) it will still be a fraction between zero and one, unless in truly catastrophic cases when $H_t > 20.58$, that is, when the temperature in period t is more than 20 degrees Celsius higher than in 1900. Of course, other forms of the damage function are possible; see [28,32,1,19]. Howard and Sterner [11] emphasize the importance of the damage function in accurately estimating coefficient and standard error bias.

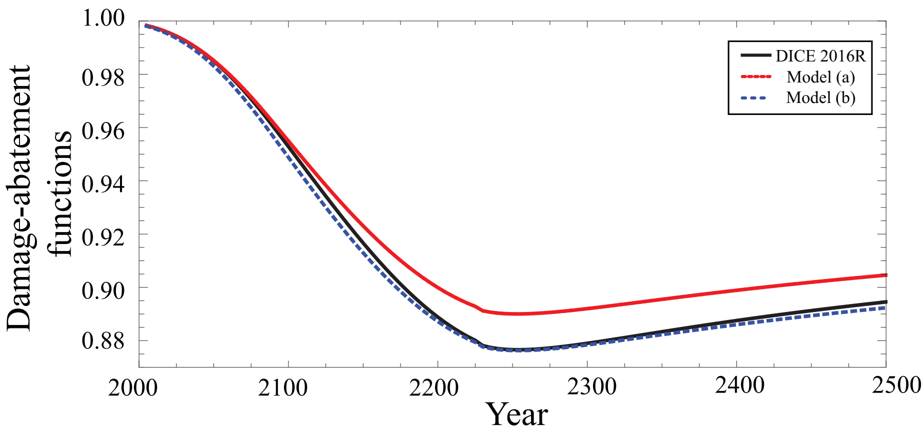

Figure 5 Damage-abatement functions. Model (a) uses the DICE value of ξ. Model (b) uses the optimal value of ξ.

In Figure 5 the graph labeled DICE 2016R contains the time path of this fraction. Models (a) and (b) use an alternative specification, namely

$$\frac{1 - \omega_t}{1 + \xi H_t^2},$$

based on the fact that the difference

$$\left(1 - \omega_t - \xi H_t^2\right) - \frac{1 - \omega_t}{1 + \xi H_t^2} = \frac{-(\omega_t + \xi H_t^2)(\xi H_t^2)}{1 + \xi H_t^2}$$

is small, about 1% in relative terms.

The alternative specification is of interest because we may wish to randomize ξ, as in [14]; see also [12,13]. This is difficult under the DICE specification because $1-\omega_t-\xi H_t^2$ could become negative (under extreme circumstances), while the alternative specification is always positive provided $\xi > 0$.

In Model (a) we use the same value for ξ as in the DICE model, while in Model (b) we use an 'optimal' value $\xi^* = 0.00265$ which brings the lines closer together; in fact, DICE and Model (b) are indistinguishable in the figure. The value ξ^* is obtained by minimizing the sum of squares

$$\sum_{t=1}^{T}\left(1-\omega_t-\xi H_t^2-\frac{1-\omega_t}{1+\xi^* H_t^2}\right)^2.$$

with respect to ξ^*. In summary, for a suitable choice of ξ we obtain an alternative for the DICE damage function which lends itself better to studying situations of uncertainty or catastrophe.

6 The S-DICE model

We summarize our proposed S-DICE (simplified DICE) model with the relevant parameters. The S-DICE model is the DICE model, but with the temperature equation replaced by the new (simplest) temperature equation

$$H_{t+1} = \eta_0^* + \eta_1^* H_t + \eta_2^* \log(M_{t+1}) \tag{22}$$

with

$$\eta_0^* = -2.8672, \quad \eta_1^* = 0.8954, \quad \eta_2^* = 0.4622,$$

and the CO_2 equation replaced by

$$M_{t+1} = \phi_1^* M_t + E_t, \qquad \phi_1^* = 0.9942. \tag{23}$$

For the damage-abatement equation we propose

$$\frac{1-\omega_t}{1+\xi^* H_t^2}, \quad \xi^* = 0.00265, \tag{24}$$

instead of the DICE specification

$$1-\omega_t-\xi H_t^2, \quad \xi = 0.00236.$$

In addition, one may wish to set $\mu \leq 1.0$ instead of the upper bound $\mu \leq 1.2$ as used in DICE. Thus, there are four differences between DICE and S-DICE.

So far we have worked within the framework of the optimized DICE model (to be precise, the baseline version with *ifopt* = 0), so that variables such as H_t and M_t take their optimal values as obtained from the DICE GAMS routine, as emphasized in Sect. 3.

To judge how the S-DICE model compares to DICE we should optimize S-DICE itself. In Table 3 we compare three scenarios: DICE and two scenarios of S-DICE, namely (a) where we use the new temperature equation (22) and the CO_2 equation (23), but not the new damage-abatement equation, while also keeping

Table 3 DICE versus S-DICE (a) and (b). Model (a) uses the DICE value of ξ. Model (b) uses the optimal value of ξ.

	Short term					Longer term		
	2020	2030	2040	2050	2060	2100	2150	2200
Temperature H								
DICE	1.0	1.4	1.7	2.1	2.5	4.1	5.7	6.7
(a)	1.0	1.4	1.8	2.3	2.7	4.5	6.4	7.6
(b)	1.0	1.4	1.8	2.1	2.5	3.6	4.1	4.1
Capital K								
DICE	268	375	505	660	840	1830	3691	6528
(a)	267	373	502	655	833	1802	3583	5968
(b)	268	374	504	660	842	1857	3884	6871
CO_2 concentration M								
DICE	891	978	1073	1177	1287	1760	2306	2649
(a)	898	1006	1129	1266	1416	2078	2859	3346
(b)	889	972	1059	1145	1226	1379	1304	1231
Abatement fraction μ								
DICE	0.03	0.04	0.04	0.05	0.06	0.11	0.24	0.53
(a)	0.03	0.04	0.04	0.05	0.06	0.11	0.24	0.53
(b)	0.23	0.29	0.36	0.44	0.54	1.00	1.00	1.00
Consumption C								
DICE	465	643	857	1105	1390	2881	5494	8860
(a)	465	643	855	1103	1385	2847	5335	8433
(b)	464	641	851	1096	1376	2854	5718	9718
Social cost of carbon SCC								
DICE	37	52	69	90	115	255	525	915
(a)	50	68	90	116	146	313	633	1089
(b)	53	72	96	124	159	363	814	1488

the 1.2 bound for μ as in DICE; and (b) where we use the full S-DICE model with all four changes. We present these results for the short term (up to 40 years) and the longer term (100–200 years).

The most important conclusion from Table 3 is that the difference between DICE and version (a) of S-DICE is small. This is true for the short and 'longer' term (up to 200 years) and remains true for the very long term (up to 500 years), as the three figures below will demonstrate. Hence, the idea that simplifications of DICE tend to have poor long-term properties is not true, at least when we consider simplifications of the temperature and CO_2 equations, which is the main topic of the current paper. In fact, the comparisons we report here give an upper bound on the approximation error of S-DICE to DICE: we could have further reduced the approximation error by calibrating the S-DICE parameters so that optimal S-DICE is as close as possible to optimal DICE.

When we add two further changes, namely the damage-abatement function and the upper bound on μ, then we obtain version (b) of S-DICE. Here the CO_2 concentration and hence the temperature is much lower than in DICE and S-DICE(a), because the new damage-abatement function of scenario (b) leads to a rapid increase in μ. In order to better understand the reason behind this difference we study below not only versions (a) and (b) but also the two intermediate versions where only one of the additional changes is implemented.

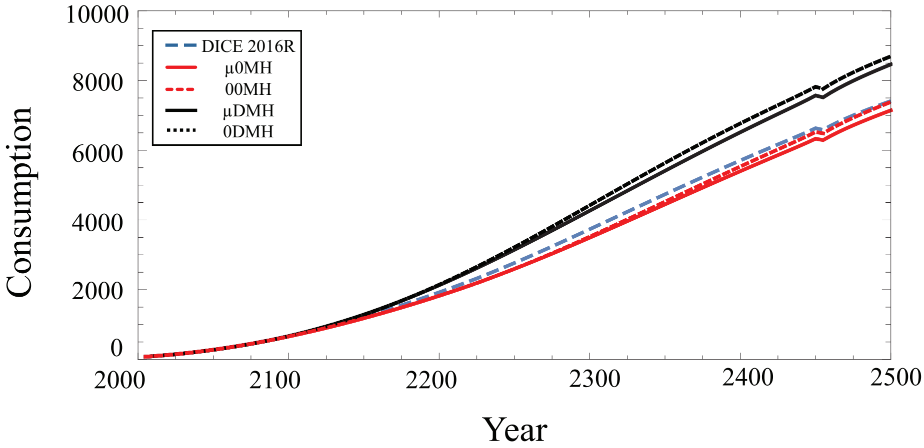

Figure 6 Consumption — DICE and S-DICE compared. Refer to the main text for the meaning of scenario acronyms such as *μDMH*.

We denote by *μDMH* the scenario where all four changes have been implemented: μ for the 1.0 upper bound on μ, D for the damage-abatement function, M for the CO_2 concentration equation, and H for the temperature equation. A zero indicates that this change has not been implemented. Hence, DICE is 0000, model (a) is *00MH* and model (b) is *μDMH*. To these three scenarios we now add the intermediate scenarios *μ0MH* and *0DMH*.

In Fig. 6 we present the time paths of consumption of the five scenarios. Consumption (and, similarly, capital accumulation) is not much affected by the different scenarios. The simplified model *00MH* with the original damage-abatement function and constraint on μ yields similar time paths to DICE, and even the full S-DICE model *μDMH* remains close to the DICE results, even in the long run. The consumption paths with a positive emissions constraint are lower than the paths without the constraint. This is because the resource allocation to abatement is larger when negative emissions are not allowed.

In the full S-DICE scenario, the new damage-abatement function causes a rapid reduction of CO_2 in the short term. In fact, μ reaches its maximum ($\mu = 1$) in 2100 when CO_2 concentration, and hence temperature, reaches its highest value. After 2100, CO_2 concentration and temperature gradually decline. Although μ reaches its maximum, consumption becomes larger than under the original damage-abatement function, due to smaller damage.

In Fig. 7 we present the time paths of temperature. With the old damage-abatement function, the time paths are close. Hence the different behavior is not caused by the constraint on μ but by the form of the damage-abatement fraction. If we include the new damage-abatement fraction but keep the constraint on μ at 1.2 (*0DMH*), so that we allow negative emissions, then the difference with

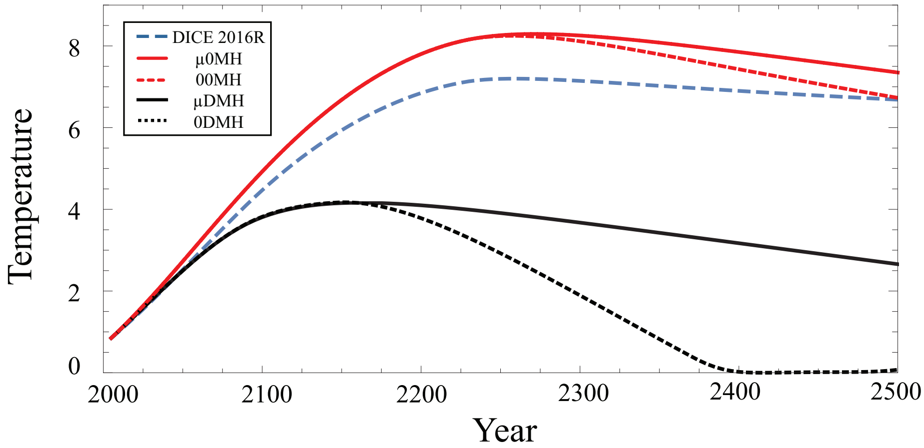

Figure 7 Temperature — DICE and S-DICE compared. Refer to the main text for the meaning of scenario acronyms such as μDMH.

DICE is much larger than if we include all four changes (μDMH). When the modified damage-abatement function and the constraint of positive emissions are both used, then the positivity constraint on temperature increase imposed by Nordhaus' GAMS code is binding from the year 2400 onwards, which affects emissions and causes a higher abatement rate and a lower level of CO_2 emissions in the short term. The figure for CO_2 concentration is similar to temperature.

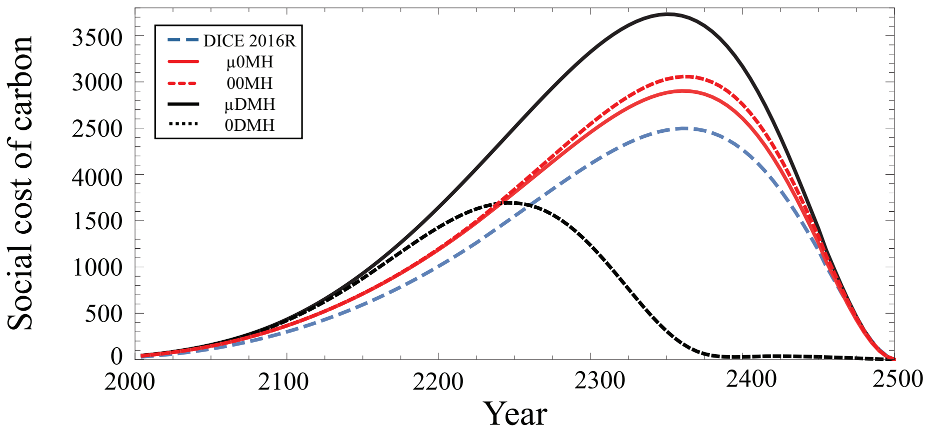

Figure 8 Social cost of carbon — DICE and S-DICE compared. Refer to the main text for the meaning of scenario acronyms such as μDMH.

Finally, we present the social cost of carbon (SCC), which attempts to answer the question how much we should be willing to pay to avert future climate damages. More precisely, the SCC tries to add up all the quantifiable costs and benefits of emitting one additional tonne of CO_2. This value can then be used to weigh the benefits of reduced warming against the costs of cutting emissions.

In Fig. 8 we present the SCC for the same five scenarios as before. We see that when negative emissions are allowed the social cost of carbon is lower than when negative emissions are not allowed. With the original damage-abatement function and the simplified temperature and CO_2 equations, the SCC paths are similar to DICE, but the new damage-abatement function leads to a higher SCC in the short run so as to reduce CO_2 rapidly. Thus, when negative emissions are not allowed, the social cost of carbon becomes much higher.

From the tables and figures we draw the following general conclusions. First, replacing the temperature and CO_2 equations by two simpler, more transparent, more robust, and easier to interpret equations does not much affect the DICE output, not even in the long run. Second, the optimized results are sensitive to the form of the damage-abatement function, especially if we allow for negative emissions. Third, the full S-DICE model leads to optimized values that are different from DICE, but within the bounds of reason. After all, there is no reason to believe that the DICE model is the truth. If S-DICE deviates from DICE then this does not necessarily mean that S-DICE is further from the truth than DICE or is less useful as a policy instrument. As noted in the introduction, there is value in simplicity. Models should be, as Einstein put it, 'as simple as possible but not simpler'.

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7 Declarations

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7.2 Conflict of interest

The authors have no relevant financial or non-financial interests to disclose.

7.3 Availability of data and material

Not applicable.

7.4 Code availability

Code is available upon request.

Appendix

In this Appendix we present two tables which together contain all variable and parameter definitions required to compute the optimum in DICE (the Nordhaus model) and S-DICE (our simplified version of DICE): the variables employed in S-DICE and DICE and their relationship, and the initial values of the six state variables of DICE (Table 4); and the parameters employed in S-DICE and DICE and their relationship (Table 5).

Table 4 Variables in S-DICE and DICE

S-DICE	DICE	Initial values
State variables		
H_t	TATM(t)	$H_1 = 0.85$
K_t	K(t)	$K_1 = 223$
M_t	MAT(t)	$M_1 = 851$
Policy variables		
I_t	$tstep \times$ I(t)	
μ_t	MIU(t)	
C_t	$tstep \times$ C(t)	
Exogenous variables		
A_t	$tstep \times$ AL(t)	
E_t^0	$tstep \times$ ETREE(t)/3.666	
F_t	$-a_1 \log(588) + 0.1005 \times$ FORCOTH(t)	
L_t	L(t)/1000	
ψ_t	COST1(t)	
σ_t	SIGMA(t)/3.666	
Auxiliary variables		
E_t	$tstep \times$ E(t)/3.666	
Y_t	$tstep \times$ YGROSS(t)	
ω_t	defined in (7)	
$X_{1,t}$	MU(t)	$X_{1,1} = 460$
$X_{2,t}$	ML(t)	$X_{2,1} = 1740$
Z_t	TOCEAN(t)	$Z_1 = 0.0068$

Table 5 Parameters in S-DICE and DICE

S-DICE	DICE	Value	Description
Basic parameters			
	a1	0	
ξ	*a2*	0.00236	
	a3	2	
	tstep	5	
	dk	0.1	
δ		0.40951	$1-\delta=(1-dk)^{tstep}$
	prstp	0.015	
ρ		0.077284	$1+\rho=(1+prstp)^{tstep}$
θ	*expcost2*	2.6	
γ	*gama*	0.3	
α	*elasmu*	1.45	
a_0		0.128189	$a_0=(c1)(fco22x)/(t2xco2)+(c1)(c3)$
a_1		0.533755	$a_1=(c1)(fco22x)/\log 2$
a_2	$c1\times c3$	0.008844	
a_3	*c4*	0.025	
b_0	*b12*	0.12	
b_1	*b21*	0.196	
b_2	*b32*	0.001465	
b_3	*b23*	0.007	
Additional parameters			
η_0		−3.329093	
η_1		0.846811	$\eta_1=1-a_0-a_3$
η_2		0.002984	$\eta_2=(a_0-a_2)a_3$
η_3		0.533755	$\eta_3=a_1$
η_4		0.013344	$\eta_4=a_1a_3$
ϕ_0		0.876134	
ϕ_{00}		263.283680	
ϕ_1		0.675535	$\phi_1=1-b_0-b_1-b_2-b_3$
ϕ_2		0.001303	$\phi_2=b_0b_2+b_0b_3+b_1b_2$
λ_1		0.795535	$\lambda_1=1-b_1-b_2-b_3$
λ_2		0.000287	$\lambda_2=b_1b_2$

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